

High-girth near-Ramanujan graphs with lossy vertex expansion

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Abstract

Kahale [Kah95] proved that linear sized sets in d -regular Ramanujan graphs have vertex expansion $\sim \frac{d}{2}$ and complemented this with construction of near-Ramanujan graphs with vertex expansion no better than $\frac{d}{2}$. However, the construction of Kahale encounters highly local obstructions to better vertex expansion. In particular, the poorly expanding sets are associated with short cycles in the graph. Thus, it is natural to ask whether *high-girth* Ramanujan graphs have improved vertex expansion. Our results are two-fold:

1. For every $d = p + 1$ for prime p and infinitely many n , we exhibit an n -vertex d -regular graph with girth $\Omega(\log_{d-1} n)$ and vertex expansion of sublinear sized sets bounded by $\frac{d+1}{2}$ whose nontrivial eigenvalues are bounded in magnitude by $2\sqrt{d-1} + O\left(\frac{1}{\log n}\right)$.
2. In any Ramanujan graph with girth $C \log n$, all sets of size bounded by $n^{0.99C/4}$ have vertex expansion $(1 - o_d(1))d$.

The tools in analyzing our construction include the nonbacktracking operator of an infinite graph, the Ihara–Bass formula, a trace moment method inspired by Bordenave’s proof of Friedman’s theorem [Bor19], and a method of Kahale [Kah95] to study dispersion of eigenvalues of perturbed graphs.

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1 Introduction

This paper is concerned with expander graphs, which are ubiquitous in theoretical computer science. A natural and highly well-studied quantity associated with a d -regular graph is its *edge expansion* defined as

$$\min_{|S| \leq \epsilon n} E(S, \bar{S})/|S|,$$

namely the minimum ratio of edges leaving a set S to the size of S for all S of appropriately bounded size. While edge expansion is known to be intractable to compute, there are explicit constructions of good edge expanders, and it is closely related to the second largest magnitude eigenvalue of its adjacency matrix, also known as *spectral expansion* of a graph, via the expander mixing lemma and Cheeger’s inequality [Alo86]. Spectral expansion is easily computable. In particular, an application of the expander mixing lemma proves that small enough sets in graphs with spectral expansion $o(d)$ have near-optimal edge expansion of $(1 - o_d(1))d$.

A natural analog to edge expansion is *vertex expansion*, defined as

$$\min_{|S| \leq \epsilon n} |\Gamma(S)|/|S|$$

for some constant ϵ , where $\Gamma(S)$ is the neighborhood of the set S (potentially containing vertices of S). However, as difficult as edge expansion is to ascertain, vertex expansion has proven far more challenging.

As witnessed by balls around a vertex, we cannot hope for vertex expansion greater than $d - 1$. Therefore we call a graph a *lossless vertex expander* if for every δ , there exists an ϵ such that there is vertex expansion $d - 1 - \delta$ for sets of size ϵn . Lossless vertex expanders exist since a random d -regular graph is one with high probability (see [HLW18, Theorem 4.16] for a proof). However no deterministic construction of such graphs is known. In an effort to understand lossless vertex expansion better and give explicit constructions, a natural question to ask is:

What properties of random graphs leads to lossless vertex expansion?

Since a random d -regular graph is near-Ramanujan with high probability [Fri03], and since near-Ramanujan graphs have near-optimal edge expansion, it is natural to inquire if spectral expansion has any implications for vertex expansion as well. Kahale [Kah95] showed that the spectral expansion gives a bound on the vertex expansion. Specifically, Ramanujan graphs (namely graphs with optimal spectral expansion) have vertex expansion at least $d/2$. While this is a nontrivial implication, it falls short of achieving the coveted *losslessness* property. Kahale also proved that the bound of $d/2$ is tight. In particular, he exhibited an infinite family of near-Ramanujan graphs with vertex expansion $d/2$, which means spectral expansion alone is not sufficient for lossless vertex expansion.

The occurrence of a copy of $K_{2,d}$ ¹ as a subgraph is the obstruction to lossless vertex expansion in Kahale’s example. Kahale’s example deviates from a random graph in that it is highly unlikely for a random graph to contain a copy of $K_{2,d}$ as a subgraph. More generally, random graphs have the property that with high probability any two “short” cycles are far apart, which Kahale’s example doesn’t satisfy. Thus, it is natural to ask if the “near-Ramanujan” property in conjunction with the “separatedness of cycles” property of random graphs break past the $d/2$ barrier of Kahale. The “separatedness of cycles” property is especially interesting to consider since it is a key property of random graphs exploited in proofs of Alon’s conjecture [Fri03, Bor19]. A concrete question we can ask is:

¹complete bipartite graph with 2 vertices on one side and d vertices on the other

Do Ramanujan graphs with $\Omega(\log_{d-1} n)$ girth have lossless vertex expansion?

An affirmative answer to the above question would prove that the Ramanujan graphs of Lubotzky, Phillips, and Sarnak [LPS88] are lossless vertex expanders. Towards answering the above question, we prove the following negative result:

Theorem 1.1. *For every $d = p + 1$ for prime p , there is an infinite family of d -regular graphs G on n vertices of girth $\geq (\frac{2}{3} - o(1)) \log n$ where there is a set of vertices U such that $|\Gamma(U)| \leq (d + 1)|U|/2$, $|U| \leq n^{1/3}$, and $\max\{\lambda_2(G), -\lambda_n(G)\} \leq 2\sqrt{d-1} + O(1/\log n)$.*

We also complement the above with a positive result which can be summarized as “small enough sets in Ramanujan graphs expand nearly losslessly”:

Theorem 1.2. *Let G be a d -regular Ramanujan graph with girth $C \log_{d-1} n$, then every set of S of size $\leq n^\kappa$ for $\kappa < \frac{C}{4}$ has vertex expansion $(1 - o_d(1))d$.*

1.1 Technical overview

We give a brief description of how [Theorem 1.1](#) and [Theorem 1.2](#) are proved.

Overview of proof of [Theorem 1.1](#). Our proof is inspired by that of Kahale’s. At a high level, Kahale embeds a copy of $K_{2,d}$ within a Ramanujan graph. We proceed similarly to Kahale, but instead of embedding a $K_{2,d}$, we embed a single subgraph H that is high girth but a lossy vertex expander and show that if H has size n^α for some $0 < \alpha \leq 1/3$, the overall graph is still near-Ramanujan.

Our proof involves two steps: the first step is in proving that the subgraph H being embedded has spectral radius bounded by $2\sqrt{d-1}$, and the second step is in proving that planting H within a Ramanujan graph results in a near-Ramanujan graph. For the first step, we describe an infinite graph containing H and bound its spectral radius via a trace moment method. The trace moment method involves bounding the number of closed walks satisfying certain properties within a graph, and is inspired by an encoding argument from Bordenave’s proof of Friedman’s theorem [Bor19].

The second step is in proving that our method of embedding a copy of H within a Ramanujan graph does not perturb the eigenvalues by a large amount. Towards doing so, we use the fact that the spectral radius of H is bounded by $2\sqrt{d-1}$ in conjunction with Kahale’s argument about dispersion of eigenvalues in high-girth graphs.

Overview of proof of [Theorem 1.2](#). We first prove that if a set S in a Ramanujan graph has “lossy” vertex expansion, then we can construct a graph H on vertex set S such that (i) the girth of H is at least half the girth of G , and (ii) the average degree of H is “high” (in particular, the worse the vertex expansion of S , the higher the average degree of H). We then employ the irregular Moore bound, which gives a quantitative tradeoff between the average degree of a graph and its girth. In particular, this would imply that a Ramanujan graph with “lossy” vertex expansion necessarily must have “low” girth.

1.2 Related work

Applications of vertex expanders. There are many applications of expander graphs where having vertex expansion is particularly useful. For example, lossless expanders are particularly of interest in the field of error correcting codes [LMSS01, SS96, Spi96]. Lossless vertex expanders give linear error

correcting codes that are decodable in linear time [SS96]. Guruswami, Lee and Razborov [GLR08] use bipartite vertex expanders to construct large subspaces of \mathbb{R}^n where all vectors x in the subspace satisfy $(\log n)^{-O(\log \log \log n)} \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$.

Explicit constructions. Constructions of Ramanujan graphs of [LPS88, Mar88, Mor94] of all degrees that are of the form $p^r + 1$ for p prime, as well as the construction of near-Ramanujan graphs of every degree of [MOP20] have vertex expansion $\sim \frac{d}{2}$ just by virtue of being Ramanujan via Kahale’s result. In fact no deterministic construction has improved upon the $d/2$ bound obtained from solely spectral information. In a remarkable work, Capalbo et. al. [CRVW02] exhibited an explicit construction of a bipartite graph where subsets of one side of the bipartite graph expand losslessly to the other, using a zig-zag product so the the losslessness of a small, random-like graph boosts the expansion from a large, potentially lossy vertex expanding graph.

Quantum Ergodicity. Quantum ergodicity is another area where both local and global properties of random-like graphs are used. In particular, Anantharaman and Le Masson [ALM15] proved that graphs that have few short cycles (and are therefore close to high girth) and spectral expansion are quantum ergodic, which in this context means the eigenvectors are equidistributed across vertices. Anantharaman, as well as Brooks, Le Masson, and Lindenstrauss exhibited alternative proofs [Ana15, BLML16]. The proof from [BLML16] shows that quantum ergodicity is equivalent to the mixing of a certain graphical operator. They then use high girth to show that this is equivalent to showing mixing on the infinite tree, then expansion to show the nonbacktracking operator mixes on the tree.

Eigenvector delocalization. Ganguly and Srivastava, and later Alon, Ganguly and Srivastava [GS18, AGS19] give a perturbation of the LPS graph similar to Kahale’s argument, but instead of individual vertices, two trees are added and connected to the graph. By assuming the tree is sufficiently deep and carefully connecting the tree to the rest of the graph, the authors create a graph that is high girth but contains eigenvectors that are localized. These graphs are also lossy vertex expanders. However, they show that these graphs cannot be Ramanujan, but rather have spectral radius at least $(2 + c)\sqrt{d - 1}$ where $c > 0$ is a constant. Alon [Alo20] used eigenvector delocalization to create near-Ramanujan expanders of every degree by perturbing known constructions of Ramanujan or near-Ramanujan graphs. Paredes [Par20] used similar techniques to remove short cycles in a graph while preserving expansion and uses this to algorithmically create graphs that are near-Ramanujan and also have girth at least $\Omega(\sqrt{\log n})$.

Complexity of constraint satisfaction problems. Proofs that it is hard for even linear degree Sum-of-Squares to refute random 3XOR and 3SAT instances on n variables [Gri01, Sch08] rely on lossless vertex expansion of some sets in a graph underlying a random instance, which suggests a connection between deterministic algorithms for constructing lossless vertex expanders and algorithms for explicit hard instances for Sum-of-Squares.

2 Preliminaries

2.1 Elementary graph theory

Definition 2.1. The *girth* $g(G)$ of a graph G is the length of the smallest cycle in G .

Definition 2.2. For $G = (V, E)$, the *valency* of $a \in V$ to $B \subset V$ is $|\Gamma(a) \cap B|$, where $\Gamma(S)$ for $S \subset V$ is the set of neighbors of S in G .

Definition 2.3. The *ball* of radius h around a set $U \subset V$, denoted $\text{Ball}_h(U)$, is the set of vertices of distance at most h from U .

Definition 2.4. The *vertex expansion* of a set $U \subset V$ is

$$\Psi(U) := \frac{|\Gamma(U)|}{|U|}.$$

Similarly, the ϵ -vertex expansion of a graph G is:

$$\Psi_\epsilon(G) = \min_{|U| \leq \epsilon|V|} \Psi(U)$$

where U ranges over subsets of V , and ϵ is an arbitrary constant.

Definition 2.5. Given a graph G , we use A_G to denote its adjacency matrix. When G is a finite graph on n vertices, the eigenvalues of A_G can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$.

Definition 2.6. We use B_G to denote the *nonbacktracking matrix* of a graph G which is a matrix with rows and columns indexed by directed edges of G defined as follows:

$$B[(u, v), (w, x)] = \begin{cases} 1 & \text{if } v = w \text{ and } u \neq x \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.7. The *spectral expansion* of a finite graph G , denoted $\lambda(G)$ is defined as $\max\{\lambda_2(G), -\lambda_n(G)\}$, which can equivalently be described as the “second largest absolute eigenvalue”.

We now state the following standard fact known as the *expander mixing lemma* (see [HLW18, Lemma 2.5]).

Lemma 2.8 (Expander Mixing Lemma). *Let G be a d -regular graph on n vertices. For any two subsets of vertices, $S, T \subseteq V(G)$, let $e(S, T)$ be the number of pairs of vertices (x, y) such that $x \in S, y \in T$ and $\{x, y\}$ is an edge in G . Then:*

$$\left| e(S, T) - \frac{d}{n} |S| \cdot |T| \right| \leq \lambda(G) \sqrt{|S| \cdot |T|}.$$

And finally, we state the “irregular Moore bound” of [AHL02] which articulates a tradeoff between the average degree of a graph and its girth.

Lemma 2.9. *Let G be a n -vertex graph with average degree- d . Then*

$$g(G) \leq 2 \log_{d-1} n + 2.$$

2.2 Operator theory

In this section, let V be a countable set and $T : \ell_2(V) \rightarrow \ell_2(V)$ be a bounded linear operator.

Definition 2.10. The *spectrum* of T , which we denote $\text{spec}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda \mathbb{1} - T$ is not invertible.

Definition 2.11. The *spectral radius* of T , which we denote $\rho(T)$ is defined as $\sup\{|\lambda| : \lambda \in \text{spec}(T)\}$.

Fact 2.12. The operator norm of T , which we write as $\|T\|$ is equal to $\sqrt{\rho(TT^*)}$ where T^* is the adjoint of T .²

Fact 2.13. $\rho(T) = \lim_{\ell \rightarrow \infty} \|T^\ell\|^{1/\ell}$.

Fact 2.14 (Consequence of [Que96, Theorem 6]). Suppose T is a self-adjoint operator, and Φ is a basis of $\ell_2(V)$. Then:

$$\rho(T) = \sup_{\phi \in \Phi} \limsup_{k \rightarrow \infty} |\langle \phi, T^k \phi \rangle|^{1/k}.$$

Fact 2.15. Let A be any principal submatrix of T . Then $\rho(A) \leq \rho(T)$.

Corollary 2.16. If H is a subgraph of (possibly infinite) graph G , then $\rho(A_H) \leq \rho(A_G)$.

3 Infinite trees hanging from a biregular graph

Let H be any $(2, d-1)$ -biregular graph where the partition with degree- $(d-1)$ vertices is called U and the partition with degree-2 vertices is called V . Let X be the infinite graph constructed from H in the following way:

At every vertex in U , the $(d-1)$ -regular partition, glue an infinite tree where the root has degree-1 and the remaining vertices have degree- d . At every vertex in V , the 2-regular partition, glue an infinite tree where the root has degree- $(d-2)$ and every other vertex has degree- d .

Note that X is a d -regular infinite graph. The main result of this section is:

Lemma 3.1. $\rho(A_X) \leq 2\sqrt{d-1}$.

To prove [Lemma 3.1](#), we instead turn our attention to the nonbacktracking matrix of X , called B_X . In particular, we bound $\rho(B_X)$ and then employ the Ihara–Bass formula of [AFH15] for infinite graphs to translate the bound on $\rho(B_X)$ into a bound on $\rho(A_X)$.

Thus, we first prove:

Lemma 3.2. $\rho(B_X) \leq \sqrt{d-1}$.

We use the following version of the Ihara–Bass formula of [AFH15] for infinite graphs.

Theorem 3.3. Let G be a (possibly infinite) graph. Then

$$\text{spec}(B_G) = \{\pm 1\} \cup \{\lambda : (D_G - \mathbb{1}) - \lambda A_G + \lambda^2 \mathbb{1} \text{ is invertible}\}.$$

An immediate corollary that we will use is:

Corollary 3.4. Let G be a d -regular graph. Then $\rho(B_G) \leq \sqrt{d-1}$ implies that $\rho(A_G) \leq 2\sqrt{d-1}$.

In light of [Corollary 3.4](#), we see that [Lemma 3.2](#) implies [Lemma 3.1](#).

Towards proving [Lemma 3.2](#), we first make a definition.

Definition 3.5. We call a walk W a $(a \times b)$ -linkage if it can be split into a segments, each of which is a length- b nonbacktracking walk.

²Since $\ell_2(V)$ comes equipped with the inner product $\langle f, g \rangle := \sum_{v \in V} f(v)g(v)$, T^* is simple the “transpose” of T .

Proof of Lemma 3.2. The first ingredient in the proof is the fact that for any $\ell \geq 0$,

$$\rho(B_X)^\ell \leq \|B_X^\ell\|$$

and thus

$$\rho(B_X) \leq \limsup_{\ell \rightarrow \infty} \|B_X^\ell\|^{1/\ell}$$

Since $\|B_X^\ell\| = \sqrt{\|B_X^\ell(B_X^*)^\ell\|} = \sqrt{\rho(B_X^\ell(B_X^*)^\ell)}$ it suffices to bound $\rho(T)$ where $T := B_X^\ell(B_X^*)^\ell$ is a bounded self-adjoint operator, and hence by [Fact 2.14](#):

$$\rho(T) = \max_{uv \in \vec{E}(X)} \limsup_{k \rightarrow \infty} |\langle 1_{uv}, T^k 1_{uv} \rangle|^{1/k}.$$

The quantity $\langle 1_{uv}, T^k 1_{uv} \rangle$ is bounded by the number of $(2k \times (\ell + 1))$ -linkages that start and end at vertex u , which we can bound via an encoding argument. In particular, we will give an algorithm to uniquely encode such linkages and bound the total number of possible encodings.

Encoding linkages. Each length- $(\ell + 1)$ nonbacktracking segment can be broken into 3 consecutive phases (of which some can possibly be empty): the phase where distance to H decreases on each step (Phase 1), the second phase where distance to H does not change on each step (Phase 2), and the third phase where distance to H increases on each step (Phase 3). We further break the third phase into two (possibly empty) subphases – the first subphase where the distance to u decreases on each step (Phase 3a), and the second subphase where the distance to u increases on each step (Phase 3b).

To encode the linkage, for each length- $(\ell + 1)$ nonbacktracking we specify four numbers denoting the lengths of Phases 1, 2, 3a, and 3b. Note that Phase 2 is nonempty only if it is contained in H . For each step ab in Phase 2 that goes from U (the $(d - 1)$ -regular partition) to V (the 2-regular partition) we specify a number i in $[d - 1]$ such that b is the i th neighbor of a within H . If the first step ab in Phase 2 is from V to U we specify a number in $[2]$ denoting if b is the first or second neighbor of a . For each step ab in Phase 3b we specify a number i in $[d - 1]$ such that b is the i th neighbor of a that does not lie in the path between u and H .

Recovering linkages from encodings. We recover a linkage from its encoding “segment-by-segment”. Suppose the first t segments have been recovered, we show how to recover the $(t + 1)$ -th segment. Let x be the vertex the walk is at after it has traversed the first t segments. The steps taken in Phase 1 can be recovered from the length of the Phase since there is a unique path from any vertex to H . The steps in Phase 2 alternate between stepping from V to U and from U to V . It is easy to recover the first step of Phase 2 as well as any step from U to V ; a step ab from V to U that is not the first step of Phase 2 is uniquely determined by the previous step, since a has 2 neighbors in U and by the nonbacktracking nature of the walk there is only one choice for b . Note that Phase 3a is nonempty only if u is not in H and all the steps are contained in the same branch as u . Since there is a unique shortest path between the start vertex of Phase 3a and u , the steps taken in Phase 3a can be recovered from its length. Finally, it is easy to recover the steps taken in Phase 3b since they are explicitly given in the encoding.

Counting encodings. Now we turn our attention to bounding the total number of encodings. For given $\alpha, \beta \geq 0$ such that $\alpha + \beta = 2k(\ell + 1)$ we first bound the number of walks such that α steps occur in Phase 2 (i.e. are within H) and β steps occur outside Phase 2 (i.e. are outside H). Let $v_1, v_2, \dots, v_{2k(\ell+1)}$ be the sequence of

vertices visited by the walk in order. Since $d(v_1, H) = d(v_{2k(\ell+1)}, H) = 0$, $|d(v_i, H) - d(v_{i+1}, H)| \leq 1$ always and $|d(v_i, H) - d(v_{i+1}, H)| = 0$ for every step in Phase 2, the number of steps of the walk that occur in Phase 3 of their respective segments is at most $\frac{\beta}{2}$. In particular, the number of steps that occur in Phase 3b of their respective segments is bounded by $\frac{\beta}{2}$. The following bounds hold:

- The number of possible encodings of the lengths of phases is bounded by $(\ell + 1)^{8k}$.
- The number of possible encodings of the first step of Phase 2 of each segment is bounded by 2^{2k} .
- The number of possible encodings of the list of U -to- V steps in Phase 2 is bounded by $(d - 1)^{\frac{\alpha+1}{2}}$ because the steps taken in Phase 2 alternate between going from V to U and from U to V .
- The number of possible encodings of the list of steps in Phase 3b is bounded by $k(\ell + 1)(d - 1)^{\frac{\beta}{2}}$.

The above bounds combined with the fact that there are at most $2k\ell$ choices for (α, β) pairs gives a bound on the number of $(2k \times (\ell + 1))$ -linkages of

$$2(k(\ell + 1))^2(\ell + 1)^{8k}2^{2k}(d - 1)^{\frac{\alpha+1}{2}}(d - 1)^{\frac{\beta}{2}} \leq 2(k(\ell + 1))^2(\ell + 1)^{8k}2^{2k}\sqrt{d - 1}^{2k(\ell+1)+1}.$$

Thus,

$$\rho(T) \leq \limsup_{k \rightarrow \infty} \left(2(k(\ell + 1))^2(\ell + 1)^{8k}2^{2k}\sqrt{d - 1}^{2k(\ell+1)+1} \right)^{1/k} = 4(\ell + 1)^8\sqrt{d - 1}^{2(\ell+1)}$$

Consequently,

$$\rho(B_X) \leq \limsup_{\ell \rightarrow \infty} \rho(T)^{1/2\ell} \leq \limsup_{\ell \rightarrow \infty} \left(4(\ell + 1)^8\sqrt{d - 1}^{2(\ell+1)} \right)^{1/2\ell} = \sqrt{d - 1}.$$

□

4 High-girth near-Ramanujan graphs with lossy vertex expansion

We will plant a high girth graph with low spectral radius within a d -regular Ramanujan graph. We will show that such a construction is a spectral expander, but has low vertex expansion. By $u \sim_G v$, we mean that u and v are adjacent in the graph G . We will write $u \sim v$ when the graph is clear from context.

Consider a $(2, d - 1)$ biregular bipartite graph $H = (U, V, E)$, with vertex components U and V . U is the degree- $(d - 1)$ component and V the degree-2 component. Therefore if we define $\gamma := |U|$, then $|V| = (d - 1)\gamma/2$. For now, we just require γ to be even. Call the vertices of U and V $\{u_1, \dots, u_\gamma\}$ and $\{v_1, \dots, v_{\gamma(d-1)/2}\}$, respectively. We connect U and V in such a way to maximize the girth of H .

Lemma 4.1.

$$g(H) \geq 2 \log_{d-1} \gamma.$$

Proof. Because of the valency conditions on H , there is a graph \tilde{H} on γ vertices $\{\tilde{u}_1, \dots, \tilde{u}_\gamma\}$, where $\tilde{u}_i \sim \tilde{u}_j$ if and only if $\exists v_k \in H$ such that $u_i \sim v_k$ and $u_j \sim v_k$. Namely, U corresponds to the vertex set of \tilde{H} , and V corresponds to the edge set. \tilde{H} is $d - 1$ regular, and as paths in \tilde{H} of length r correspond to paths of length $2r$ in H , $g(H) = 2g(\tilde{H})$.

By a result of Linial and Simkin [LS19], there exists a graph \tilde{H} that has girth at least $c \log_{d-2} \gamma$, for any $c \in (0, 1)$, assuming γ is even. Therefore by setting $c = \log(d - 2)/\log(d - 1)$, we have that $g(\tilde{H}) \geq \log_{d-1} \gamma$ and $g(H) \geq 2 \log_{d-1} \gamma$. □

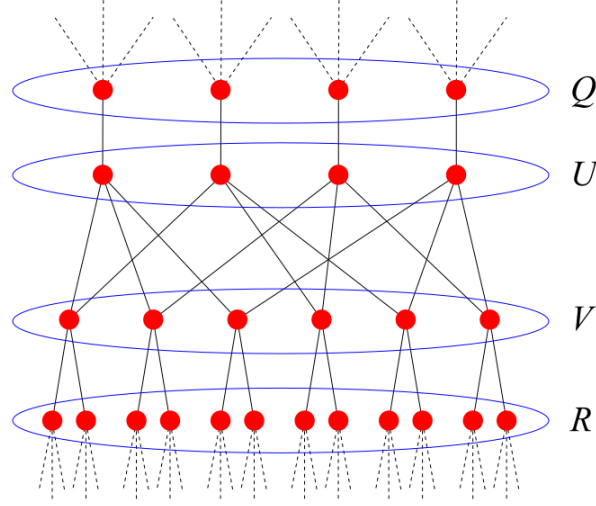


Figure 1: H' , with labeled components for $d = 4$, $\gamma = 4$. Note that $\Psi(U) = (d + 1)/2$. To create G' , we connect Q and R to a well spaced matching in G .

We now add a new set of vertices $Q = \{q_1, \dots, q_\gamma\}$ and add a matching between Q and U , adding the edge $q_i u_i$ for $1 \leq i \leq \gamma$. Similarly, we add another set of vertices $R = \{r_{i,j}, 1 \leq i \leq \gamma(d-1)/2, 1 \leq j \leq d-2\}$, and for $1 \leq i \leq \gamma(d-1)/2$ add an edge from v_i to each of $r_{i,j}$ for $1 \leq j \leq (d-2)$.

We call H' the graph on $U \cup V \cup Q \cup R$. At this point vertices of U and V have degree- d , and vertices of Q and R have degree-1. Also, note $\Psi(U) = (d + 1)/2$. We wish to embed H' into a larger, high girth expander, and show that this new graph maintains high girth and expansion, even though the set U is a lossy vertex expander. Our argument follows that of [Kah95, Section 5], but instead of embedding individual vertices, we will embed H' .

First we need a slight generalization of [Kah95, Lemma 5.1]. The result follows from the same proof, but Kahale did not need this precise statement in that argument.

Lemma 4.2 (Lemma 5.1 of [Kah95]). *Consider a graph on a vertex set W , a subset X of W , a positive integer h , and $s \in L^2(W)$. Let X_i be the set of nodes at distance i from X . Assume the following conditions hold:*

- (1) *For $h - 1 \leq i, j \leq h$, all nodes in X_i have the same number of neighbors in X_j .*
- (2) *If $u \in X_{h-1}$ and $v \in X_h$ and $u \sim v$, then the valency of u to X_h , the valency of v to X_{h-1} and $s(u)/s(v)$ do not depend on the choices of u and v .*
- (3) *s is nonnegative and $As \leq \mu s$ on $\text{Ball}_{h-1}(X)$, where μ is a positive real number.*

Then for any $g \in L^2(W)$ such that $|Ag(u)| = \mu|g(u)|$ for $u \in \text{Ball}_{h-1}(X)$, we have

$$\frac{\sum_{v \in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \geq \frac{\sum_{v \in X_{h-1}} g(v)^2}{\sum_{v \in X_{h-1}} s(v)^2}.$$

Using this, we can now state our theorem.

Theorem 4.3. *For every $d = p + 1$ for prime p , there is an infinite family of d -regular graphs $G_m = (V_m, E_m)$ on m vertices, such that $\exists U_m \subset V_m$ with $\Psi(U_m) = (d + 1)/2$ for $|U_m| \leq m^{1/3}$, $g(G_m) = (\frac{2}{3} - o(1)) \log m$, and such that $\lambda(G_m) \leq 2\sqrt{d-1} + O(1/\log m)$.*

Proof. By the famous result of Lubotzky, Phillips and Sarnak [LPS88], for infinitely many d , there exists an infinite family of d -regular graphs, where graphs of n vertices have girth $(\frac{4}{3} - o(1)) \log_{d-1} n$ and have spectral expansion $\leq 2\sqrt{d-1}$.

For a given graph $G = (V, E)$ of this type of size n , we will attach H' by removing a matching $M \subset E$, $M = \{(a_{1,1}, a_{1,2}), \dots, (a_{k,1}, a_{k,2})\}$ for

$$k = \gamma(d-1)(2 + (d-1)(d-2))/4. \quad (1)$$

We take a matching such that the pairwise distance between edges in the matching is maximized in G .

Lemma 4.4. *In a d -regular graph on n vertices, there exists a matching M such that for every pair of edges $(a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2}) \in M$, $i_1 \neq i_2$,*

$$d((a_{i_1,1}, a_{i_1,2}), (a_{i_2,1}, a_{i_2,2})) \geq \log_{d-1} n - \log_{d-1} \gamma - O(1).$$

Proof. For a given pair of adjacent vertices $(a_{i_1,1}, a_{i_1,2})$, as our graph is d regular, there are at most $1 + d \frac{(d-1)^r - 1}{d-2}$ vertices at distance at most r from $a_{i_1,1}$, and at most $(d-1)^r$ vertices at distance r from $a_{i_1,2}$ and distance $r+1$ from $a_{i_1,1}$. Therefore for any $d \geq 4$, the number of edges at distance at most r from a given edge is less than $4k(d-1)^r$. We then greedily add edges by choosing an arbitrary edge whose vertices are distance at least r away from all already chosen edges. A k th such edge will exist as long as $4k(d-1)^r \leq n$. For our k given in (1) we can set $r = \log_{d-1} n - \log_{d-1} \gamma - O(1)$. \square

To connect H' to G , we first delete the matching M . Then for every vertex of Q and R , we add $d-1$ edges to the set of vertices of M , connecting to each vertex of M exactly once. Namely, the induced subgraph on $(Q \cup R) \cup M$ is a $(d-1, 1)$ biregular bipartite graph. Call $G' = (V', E')$ the new graph formed from G and H' .

We wish to show that G' remains high girth and a good spectral expander. For the girth of G' , cycles are either completely contained in H' , completely contained in G , or a mix between the two. Cycles in H' have length at least $2 \log_{d-1} \gamma$ by Lemma 4.1. Cycles in G have length at least $(\frac{4}{3} - o(1)) \log_{d-1} n$ by the construction of [LPS88]. For cycles that are a mix of H' and G , we must go from one vertex of H' to another vertex of H' through G . Therefore by Lemma 4.4, the length of such a cycle is at least $\log_{d-1} n - \log_{d-1} \gamma - O(1)$, giving

$$g(G') \geq \min\{2 \log_{d-1} \gamma, \log_{d-1} n - \log_{d-1} \gamma - O(1)\}.$$

To show that the spectrum is not adversely affected, we follow the argument of [Kah95, Theorem 5.2], with some adjustments. For our new graph, assume that there is an eigenvector $g \perp \mathbf{1}$ corresponding to an eigenvalue $|\mu| > 2\sqrt{d-1}$.

Call A the adjacency matrix of G' , and A_G the adjacency matrix of G . Then we have

$$g^* A g = g_G^* A_G g_G + g_{H'}^* A g_{H'} - 2 \sum_{i=1}^k g(a_{i,1}) g(a_{i,2}) + \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u) g(a_{i,j})$$

where g_G and $g_{H'}$ are the projections of g onto G and H' , respectively.

We know that

$$|g_G^* A_G g_G| \leq 2\sqrt{d-1} \|g_G\|^2 + \frac{d}{n} \left(\sum_{u \in G} g(u) \right)^2$$

by decomposing our vector into parts parallel and perpendicular to the all ones vector.

By a combination of [Lemma 3.1](#) and [Corollary 2.16](#), the spectral radius of H' is $2\sqrt{d-1}$, and therefore we have

$$|g_G^* A_G g_G| + |g_{H'}^* A g_{H'}| \leq 2\sqrt{d-1} \|g\|^2 + \frac{d}{n} \left(\sum_{u \in H'} g(u) \right)^2$$

as $\sum_G g(u) = -\sum_{H'} g(u)$, considering $g \perp \mathbf{1}$.

To show that $|\mu| = 2\sqrt{d-1} + O(1/\log n)$, we then need to show

$$\frac{1}{\|g\|^2} \left(\frac{d}{n} \left(\sum_{H'} g_{H'}(u) \right)^2 - 2 \sum_{i=1}^k g(a_{i,1})g(a_{i,2}) + \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u)g(a_{i,j}) \right) = O\left(\frac{1}{\log n}\right). \quad (2)$$

The first term of (2) can be bounded as

$$\frac{d}{n} \left(\sum_{H'} g_{H'}(u) \right)^2 \leq \frac{d}{n} |H'| \|g_{H'}\|^2 \leq \frac{\gamma(2 + (d-1)(d-2))d}{2n} \|g_{H'}\|^2. \quad (3)$$

The second term we can bound as

$$\left| 2 \sum_{i=1}^k g(a_{i,1})g(a_{i,2}) \right| \leq \sum_{a_{i,j} \in M} g(a_{i,j})^2. \quad (4)$$

Now we will bound the last term of (2) using Cauchy Schwarz.

$$\left| \sum_{\substack{u \in Q \cup R \\ a_{i,j} \in M \\ u \sim a_{i,j}}} g(u)g(a_{i,j}) \right| \leq \sqrt{(d-1) \sum_{u \in Q \cup R} g(u)^2} \sqrt{\sum_{a_{i,j} \in M} g(a_{i,j})^2}. \quad (5)$$

At this point, we will use [Lemma 4.2](#) to bound (4) and (5). For the purposes of the lemma, we set $X_0 = U \cup V$, and h will vary from $2 \leq h \leq \lfloor r/2 \rfloor$. Assuming that the girth of G' is at least r , the $\lfloor r/2 \rfloor$ neighborhoods of each vertex do not overlap.

The following is motivation for the vector s that we choose for [Lemma 4.2](#). The valency from a vertex in X_0 to the set X_1 depends on the choice of vertex, and vice versa. However if we split $X_0 = U \sqcup V$ and split $X_1 = Q \cup R$, then the valency of a vertex in X_0 to X_1 only depends on whether our vertex is in U or V , and the valency of a vertex in X_1 to X_0 depends only on whether it is in Q or R . We continue this process, separating X_h into $X_{h,U}$ and $X_{h,V}$, which are the vertices that are closer to U and closer to V , respectively.

At this point, we can create a *quotient matrix*, where entry i, j corresponds to the number of neighbors a vertex $u \in X_i$ has in X_j . For an overview of quotient matrices and their utility, see Godsil, [[God93](#), Chapter 5]. [[God93](#), Corollary 5.2.3] gives that spectral radius of $\text{Ball}_h(X)$ is the same as that of the quotient matrix, and that the eigenvectors of the quotient matrix correspond to eigenvectors f of the adjacency matrix for which the value $f(u)$ is only determined by the set in the partition it is a part of. The quotient matrix for $\text{Ball}_h(X)$ is

$$\tilde{A} = \begin{array}{ccccccc} \ddots & & & & & & \\ & 0 & 1 & 0 & 0 & 0 & 0 \\ & d-1 & 0 & 1 & 0 & 0 & 0 \\ & 0 & d-1 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 & d-1 & 0 \\ & 0 & 0 & 0 & 2 & 0 & d-2 \\ & 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & \ddots \end{array} \left. \begin{array}{l} \vdots \\ X_{3,U} \\ X_{2,U} \\ X_{1,U} \\ X_{0,U} \\ X_{0,V} \\ X_{1,V} \\ X_{2,V} \\ \vdots \end{array} \right\}$$

where on the right, we have signified the set corresponding to the row of the matrix.

Based on this, we can come up with a vector s

$$s(y) = \begin{cases} \frac{1}{(d-1)^{h/2}} & X_{h,U} \\ \frac{2}{\sqrt{d-1}} - \frac{1}{(d-1)^{3/2}} & y \in X_{0,V} \\ \left(\frac{2}{d-2} - \frac{2}{(d-1)(d-2)} \right) \frac{1}{(d-1)^{(h-1)/2}} & y \in X_{h,V}, h \geq 1. \end{cases}$$

For this assignment of values we have $As \leq (2\sqrt{d-1})s$. In fact, this inequality is sharp at all coordinates except for $y \in X_{1,V}$.

For this s , we have that $\sum_{y \in X_h} s(y)^2$ is constant for $h = 1, \dots, \lfloor r/2 \rfloor$. Also, remember $Q \cup R = X_1$ and $M = X_2$. By [Lemma 4.2](#), as g corresponds to an eigenvalue $|\mu| > 2\sqrt{d-1}$, the mass on each of first 2 layers of X can only be at most $2/(r-2)$ of the total mass.

Combining [\(3\)](#), [\(4\)](#), and [\(5\)](#), we can bound [\(2\)](#) as

$$\begin{aligned} (2) &\leq \frac{\gamma(2 + (d-1)(d-2))d}{2n} \|g_{H'}\|^2 + \sum_{a \in X_2} g(a)^2 + \sqrt{(d-1) \sum_{u \in X_1} g(u)^2} \sqrt{\sum_{a \in X_2} g(a)^2} \\ &\leq \left(\frac{\gamma(2 + (d-1)(d-2))d}{2n} + (1 + \sqrt{d-1}) \frac{2}{r-2} \right) \|g\|^2. \end{aligned}$$

If we set $\gamma = n^{1/3}$ and $r = \frac{2}{3} \log_{d-1} n - O(1)$, for fixed d this becomes

$$O\left(\frac{1}{\log n}\right) \|g\|^2,$$

meaning that $\mu \leq 2\sqrt{d-1} + O(1/\log n)$. This also gives the desired bounds on vertex expansion and girth, by setting $U = U_m$. Because $|V'| = (1 + o(1))n$, the bounds on $\Psi(U_m)$, $g(G')$ and $\lambda(G')$ given in terms of n do not change when they are given in terms of m . \square

5 Lossless expansion of small sets

In this section, we prove that sufficiently small sets in a high-girth spectral expander expand losslessly.

Theorem 5.1. *Let G be a d -regular graph on n vertices with girth at least $2\alpha \log_{d-1} n + 4$. Then for any set S with n^κ vertices,*

$$\frac{|\partial S|}{|S|} \geq d - \lambda(G) - \frac{d^{\kappa/\alpha}}{2} - \frac{d}{n^{1-\kappa}}.$$

Proof. Let S be a set of vertices of size n^κ in G . Let e_S denote the number of internal edges within S . Let n_i denote the number of vertices in ∂S that have i edges from S incident to it. Then: $|\partial S| = n_1 + n_2 + \dots + n_d$ and $|E(S, \partial S)| = n_1 + 2n_2 + \dots + dn_d$. Note that $|E(S, \partial S)|$ is also equal to $d|S| - 2e_S$. Now consider the graph H_S on vertex set S and edge set given by induced edges on S along with new edges introduced by adding an arbitrary spanning tree for every set of i vertices that are neighbors of a vertex in ∂S with exactly i neighbors in S . The number of edges in $H(S)$ is equal to

$$e_S + n_2 + 2n_3 + \dots + (d-1)n_d = e_S + |E(S, \partial S)| - |\partial S| = d|S| - e_S - |\partial S|$$

and $g(H(S)) \geq \frac{1}{2}g(G) \geq \alpha \log_{d-1} n + 2$. As a consequence of the expander mixing lemma (Lemma 2.8),

$$e_S \leq \left(\lambda(G) + \frac{d|S|}{n} \right) |S|$$

for some absolute constant c . Consequently,

$$|E(H(S))| \geq \left(d - \lambda(G) - \frac{d|S|}{n} \right) |S| - |\partial S|,$$

which means the average degree is lower bounded by

$$2 \left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\partial S|}{|S|} \right).$$

Thus by the irregular Moore bound (Lemma 2.9),

$$g(H(S)) \leq \frac{2 \log n^\kappa}{\log \left(2 \left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\partial S|}{|S|} \right) - 1 \right)} + 2$$

and hence

$$\frac{\alpha}{\log(d-1)} \leq \frac{2\kappa}{\log \left(2 \left(d - \lambda(G) - \frac{d|S|}{n} - \frac{|\partial S|}{|S|} \right) - 1 \right)}.$$

This implies

$$d - \lambda(G) - \frac{d|S|}{n} - \frac{|\partial S|}{|S|} - \frac{1}{2} \leq \frac{d^{2\kappa/\alpha}}{2},$$

and finally by rearranging the above and plugging in $|S| = n^\kappa$

$$\frac{|\partial S|}{|S|} \geq d - \lambda(G) - \frac{d^{2\kappa/\alpha} - 1}{2} - \frac{d}{n^{1-\kappa}}.$$

□

Remark 5.2. If G is a n -vertex d -regular Ramanujan graph with girth $\frac{4}{3} \log_{d-1} n$ (which is a condition satisfied by the Ramanujan graphs of [LPS88]) then for every set S of size n^κ for $\kappa < 1/3$,

$$\frac{|\partial S|}{|S|} \geq d(1 - o_d(1)).$$

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