CS 496 Homework 1

Due Date: October 28

Instructions

This problemset might be harder than intended so feel free to email me to ask for hints! When you're done, email me your solutions with an email titled `CS496 HW1 solutions (your name)`.

1 Spectral graph theory basics $+\varepsilon$

Let G be a graph on n vertices, average degree d and maximum degree $d_{\max}$, and let A_G be its unnormalized adjacency matrix.

1. Prove the following inequalities:

$$\begin{aligned}
\lambda_{\max}(A_G) &\geqslant d \\
\lambda_{\max}(A_G) &\leqslant d_{\max} \\
\lambda_{\max}(A_G) &\geqslant \sqrt{d_{\max}}
\end{aligned}$$

2. Prove that there always exists an orientation of edges in G such that the out-degree of every vertex is at most $\lambda_{\max}(A_G)$.

3. Let G be a d-regular connected graph such that $\lambda_2(A_G) > \lambda_3(A_G)$. Let f_2 be the second eigenvector of G, and let G_2 be the graph obtained by deleting all edges uv such that $\operatorname{sign}(f_2(u)) \neq \operatorname{sign}(f_2(v))$. Prove that G_2 has exactly two connected components.

   Hint: *Use the variational characterization of eigenvalues and eigenvectors.*

4. Let G and H be graphs. We define the *Cartesian product* of two graphs $G \times H$ as the graph whose vertex set is $V(G) \times V(H)$, and edges described by $\{(a,b),(a,c)\}$ for $a \in V(G)$ and $\{b,c\} \in E(H)$, and $\{(a,c),(b,c)\}$ for $c \in V(H)$ and $\{a,b\} \in E(G)$.

   Prove that the eigenvalues of $A_{G\times H}$ are described by $\{\lambda + \mu : \lambda \in \operatorname{Spec}(A_G), \mu \in \operatorname{Spec}(A_H)\}$.

5. Write down all the eigenvalues of the d-dimensional Boolean hypercube.

6. Recall the notion of a 2-*lift* of a graph G from Lecture 7. Prove that the spectrum of a 2-lift G_σ of G described by the signing σ is equal to $\operatorname{Spec}(A_G) \sqcup \operatorname{Spec}(A_{G,\sigma})$ where $A_{G,\sigma}$ is the corresponding signed adjacency matrix.

2 Markov chains and eigenvalues

Let G be a graph, and let P_G be its random walk transition matrix, where $P_G[u,v] = \frac{1}{\deg(u)}$.

1. Prove that if G is connected, then it has a unique stationary distribution π.
2. Prove that π satisfies the *detailed balance condition*: i.e., for any pair of vertices u and v, $\pi(u) \cdot P_G[u,v] = \pi(v) \cdot P_G[v,u]$.
3. Suppose ν_0 is a probability distribution on $V(G)$, and ν_1 is the distribution of sampling $x \sim \nu_0$ and then taking a random step to y according to P_G. Define $f_0 = \frac{\nu_0}{\pi}$ and $f_1 = \frac{\nu_1}{\pi}$. Prove that $f_1 = P_G f_0$.
4. The measure of how close two probability distributions μ and ν are is the *total variation distance*, which has several equivalent definitions. Prove that all of these definitions are equal.

$$d_{\mathrm{TV}}(\mu,\nu) := \max_{\mathcal{E} \text{ events}} |\mu(\mathcal{E}) - \nu(\mathcal{E})| = \frac{1}{2}\sum_x |\mu(x) - \nu(x)| = \frac{1}{2}\mathbf{E}_{x\sim\mu}\left|1 - \frac{\nu}{\mu}(x)\right|.$$

5. Prove that $d_{\mathrm{TV}}(\mu,\nu) \leqslant \sqrt{\frac{1}{2}\mathbf{Var}_{x\sim\mu}\frac{\nu}{\mu}(x)}$.
6. Let ν_t be the distribution obtained by running a t-step random walk initialized from a vertex sampled from ν_0. Prove that if $\max\{\lambda_2(P_G), |\lambda_n(P_G)|\} < 1-\varepsilon$, then for every ν_0:

$$d_{\mathrm{TV}}(\nu_t,\pi) \leqslant \sqrt{\frac{1}{2}\cdot(1-\varepsilon)^t \cdot \frac{1}{\min_i \pi(i)}}\,.$$

3 Lossless expanders and error-correcting codes

We say that a (c,d)-biregular n-vertex bipartite graph G with left vertex set L, right vertex set R, and edge set E is an *ε-one-sided lossless expander* if there is a constant η (possibly depending on c, d and ε, but independent of n) such that for every $S \subseteq L$ with $|S| \leqslant \eta|L|$, we have:

$$|N(S)| \geqslant (1-\varepsilon)\cdot c \cdot |S|,$$

where $N(S) \subseteq R$ refers to the neighborhood of S.

1. Prove that a random (c,d)-biregular graph drawn from the configuration model is an ε-one-sided lossless expander with high probability for some $\varepsilon = o_{c,d}(1)$.
2. Let $c < d$, and consider the linear subspace $\mathcal{C} \subseteq \mathbb{F}_2^L$ of all f such that for all $v \in R$, $\sum_{u\sim_G v} f(u) = 0$. Prove that $\mathcal{C}$ is a good code.
3. Design a linear time algorithm that takes as input a "corrupted" codeword, i.e., $y \in \mathbb{F}_2^L$ such that there is an x with Hamming distance at most $\eta|L|/2$ close to y, and outputs the closest codeword x.
4. In this part, we will see how to construct a lossless expander from a good spectral expander.

   Let's take the following fact for granted: for every $K, C \in \mathbb{N}$, there is an explicit infinite family of graphs that are (K,C)-biregular, and $\lambda_2(G) \leqslant \sqrt{K-1} + \sqrt{C-1}$ for any G in this family.

Let H be a (c,d)-biregular graph with C vertices on the left and D vertices on the right that is a $o_{C,D}(1)$-lossless expander.

The routed product Z of G and H is defined by taking G, creating D copies $v^{(1)},\dots,v^{(D)}$ of every vertex v in $R(G)$, and then placing a copy of H between the C neighbors $u_1,\dots,u_C$ of v in $L(G)$ and $v^{(1)},\dots,v^{(D)}$.

Prove that Z is a lossless expander.

4 On forbidden subgraphs

Let G be a d-regular Ramanujan graph on n vertices.

1. Use the expander mixing lemma to prove that there is a constant $\eta > 0$ (possibly depending on d) such that for every set S with at most ηn vertices has vertex expansion $\approx \frac{d}{4}$. I.e. we have:
$$\frac{|N_G(S)|}{|S|} \geqslant \frac{d}{4} - o(d)$$
where $N_G(S)$ denotes the neighborhood of a set S within G.

2. Given a set S of vertices, use $G[S]$ to denote the induced subgraph of G on the vertices S. Prove that for any set S, we have:
$$\lambda_{\max}(A_{G[S]}) \leqslant \frac{d|S|}{n} + \lambda_2(A_G)\,.$$

3. Improve on the bound from part 1. and prove that:
$$\frac{|N_G(S)|}{|S|} \geqslant \frac{d}{2} - o(d)\,.$$

Hint: *I don't think this is achievable by just using the expander mixing lemma. Try the following style of proof strategy instead. Assume for contradiction that S fails to exhibit the desired vertex expansion. Consider an ℓ-ball S_ℓ around $S \cup N_G(S)$ for some large constant ℓ, and exhibit a test vector v that witnesses that the subgraph $G[S_\ell]$ has a large eigenvalue and use 2. to prove that this shows a violation of the second eigenvalue bound.*