CS 496 Lecture 10: Interlacing Polynomials and Ramanujan Graphs II

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1 Introduction and setup

Recall that last week, we covered the proof by Marcus, Spielman, and Srivastava [MSS15] of the existence of an infinite family of bipartite d-regular Ramanujan graphs for any $d \geqslant 3$, modulo two technical lemmas, which we recall here.

Let G be a d-regular n-vertex graph.

Lemma 1.1 (Interlacing family). *The family $(\chi_{G_\sigma}(x))_\sigma$ of characteristic polynomials of signings of G form an interlacing family.*

Lemma 1.2 (Root bounds for matching polynomial). *The matching polynomial $\mu_G(x)$ of G is real-rooted and all its roots are in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.*

We will omit the proof of Lemma 1.1, and refer the reader to [MSS15] for a proof, and will spend this lecture proving Lemma 1.2. This was first proved in the work of Heilmann and Lieb [HL72] who were studying the matching polynomial in the context of monomer-dimer systems from statistical physics. Later, Godsil [GG78] gave an alternate proof. The proof we will see today is one in the work of Mohanty and O'Donnell [MO20].

The fact that $\mu_G(x)$ is real-rooted follows from the fact that it is the expected characteristic polynomial of an interlacing family. We delve into bounds on the roots next.

2 Root bounds for the matching polynomial

The key in the proof we present today is to calculate moments of the roots, which we will do via a key connection between root power sums and closed treelike walks in G.

Definition 2.1 (Treelike walks). Given a walk $W = u_0u_1 \dots u_\ell u_0$, we maintain a set of active edges S_t at time t. Initially, S_0 is empty. If an edge e is used at time t, we define S_t as $S_{t-1}\Delta\{e\}$. We say that W is a treelike walk if at all time t, the set of edges S_t forms a tree.

Lemma 2.2. *Let $\gamma_1, \dots, \gamma_n$ be the roots of $\mu_G(x)$. Then the root power sums count closed treelike walks. I.e., for any $\ell \geqslant 0$:*

$$\sum_{i=1}^{n} \gamma_i^\ell = \#\,\textit{closed length-}\ell\textit{ treelike walks in } G\,.$$

We will use the above to prove:

Lemma 2.3. $\max\{|\gamma_1|, |\gamma_n|\} \leqslant \lim_{\ell\to\infty}\left(\sum_{i=1}^{n}\gamma_i^{2\ell}\right)^{1/2\ell} \leqslant 2\sqrt{d-1}$.

Proof. By Lemma 2.2, it suffices to establish a bound on the number of closed length-2ℓ treelike walks in G. For any fixed vertex v, the number of length-2ℓ closed treelike walks that start and end at v is at most the number of closed length-2ℓ walks in the d-regular infinite tree, which we established was at most $(2\sqrt{d-1})^{2\ell}$.

Since there are n choices for the starting vertex v, we see that the number of closed length-2ℓ treelike walks in G is at most

$$(2\sqrt{d-1})^{2\ell}\cdot n\,.$$

This bound holds true for any ℓ, and hence the desired statement follows. □

Note that Lemma 1.2 then follows immediately. The strategy for proving Lemma 2.2 is to use the *Newton identities*.

Lemma 2.4 (Newton identities). *For a polynomial $p(x) = x^n + \widehat{p}_{n-1}x^{n-1} + \cdots + \widehat{p}_1 x + \widehat{p}_0$ with roots $\gamma_1, \ldots, \gamma_n$, we have:*

$$k\cdot\widehat{p}_{n-k} = -\sum_{i=1}^{k}\widehat{p}_{n-(k-i)}\cdot\sum_{t=1}^{n}\gamma_t^i\,.$$

Newton's identities gives us a handle on the root moments, and the proof is by showing that one can relate the count of matchings in G with the count of treelike walks via a recursion that looks exactly like Newton's identities!

A reformulation of Newton's identities based on the fact that $\widehat{p}_{n-k} = (-1)^k\sum_{S\subseteq[n]:|S|=k}\prod_{i\in S}\gamma_i$ is given below.

Lemma 2.5 (Newton's identities, reformulated). *Let $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$, let $e_k(\vec{\gamma}) = \sum_{S\subseteq[n]:\,|S|=k}\prod_{i\in S}\gamma_i$ be the k-th elementary symmetric polynomial, and let $p_k(\vec{\gamma}) = \sum_{i=1}^{n}\gamma_i^k$ be the k-th power sum. Then for every $k \geqslant 1$,*

$$k\cdot e_k(\vec{\gamma}) \;=\; \sum_{i=1}^{k}(-1)^{i-1}\,e_{k-i}(\vec{\gamma})\,p_i(\vec{\gamma})\,.$$

Proof. Fix $k \geqslant 1$. For each $r \geqslant 1$ define the "one-repetition class"

$$J_r := \sum_{u=1}^{n}\sum_{\substack{J\subseteq[n]\setminus\{u\}\\ |J|=k-r}}\gamma_u^r\prod_{j\in J}\gamma_j.$$

Thus J_r is the sum of all degree-k monomials in which *at most one* variable is repeated, and if there is a repetition it has multiplicity exactly r.

A key observation is the following two-way decomposition, valid for every $i = 1, \ldots, k$:

$$e_{k-i}(\vec{\gamma})\,p_i(\vec{\gamma}) \;=\; J_i \;+\; J_{i+1}, \tag{1}$$

where $J_{k+1} = 0$. Indeed, expanding $e_{k-i}p_i$ produces terms

$$\Big(\prod_{j\in S}\gamma_j\Big)\,\gamma_t^i \quad \text{with } S\subseteq[n],\ |S| = k-i,\ t\in[n].$$

These split into two disjoint types:

- If $t \notin S$, the resulting monomial has a single i-th power (at t) and $k-i$ distinct other variables: this contributes to J_i.
- If $t \in S$, then t appears once from S and i more times from p_i, giving a single $(i+1)$-power and $k-(i+1)$ other distinct variables: this contributes to J_{i+1}.

Now, observe that

$$\sum_{i=1}^{k}(-1)^{i-1}\, e_{k-i}p_i = \sum_{i=1}^{k}(-1)^{i-1}\,(J_i + J_{i+1}) = J_1 + (-1)^{k-1}J_{k+1} = J_1$$

J_1 is easily seen to equal $k \cdot e_k$, which completes the proof. □

Lemma 2.6 (Matchings $\leftrightarrow$ treelike closed walks)**.** *Let G be a finite simple graph. For $k \geqslant 0$ let $m_k(G)$ be the number of k-matchings in G, and for $i \geqslant 0$ let $t_{2i}(G)$ be the number of length-$2i$ closed treelike walks in G. Then for every $k \geqslant 1$,*

$$\boxed{2km_k(G) = \sum_{i=1}^{k}(-1)^{\,i-1}\, m_{k-i}(G)\, t_{2i}(G).}$$

Equivalently, if one sets $e_{2j} = (-1)^j m_j$ and $p_{2i} = t_{2i}$ (with all odd indices 0), this is exactly Newton's recursion $re_r = \sum_{i=1}^{r}(-1)^{i-1}e_{r-i}p_i$ at $r = 2k$.

Observe that Lemma 2.2 follows from Lemma 2.6.

Proof of Lemma 2.6. Fix $k \geqslant 1$. For $i \geqslant 1$ let

$$\mathcal{A}_i := \{(M,W) : M \text{ a } (k-i)\text{-matching in } G,\ W \text{ a length-}2i \text{ treelike closed walk in } G\}.$$

Thus $|\mathcal{A}_i| = m_{k-i}(G)\, t_{2i}(G)$, and the alternating sum in the statement is

$$\sum_{i=1}^{k}(-1)^{i-1}\,|\mathcal{A}_i|.$$

Two elementary operations. For an edge uv of G and a closed walk $W = (w_0, w_1, \ldots, w_{2i} = w_0)$, let $t = t_{uv}(W)$ be the first index with $w_t \in \{u, v\}$ (if it exists). Without loss of generality, assume $w_t = u$. The *splice* of W along uv is

$$\mathrm{splice}_{uv}(W) := (w_0, \ldots, w_{t-1},\ u, v, u,\ w_{t+1}, \ldots, w_{2i})\,.$$

If W contains a backtrack $x\,y\,x$ such that x and y are both visited for the first time in this backtrack, let $\mathrm{erase}_{xy}(W)$ be the walk obtained by deleting that y and the following x; this shortens the length by 2. Both operations preserve the treelike property.

Earliest events. Given $(M,W) \in \mathcal{A}_i$, define two (discrete) times along W:

- $\tau_{\mathrm{back}}(W)$: the index of the *first* backtrack of W (necessarily exists since $i \geqslant 1$);
- $\tau_{\mathrm{touch}}(M,W)$: the first index t such that w_t is incident to some edge of M (set ∞ if no vertex of W is incident to M).

Note that if $\tau_{\text{back}}(W) < \tau_{\text{touch}}(M, W)$ then the backtrack at time τ_{back} uses an edge uv whose endpoints are disjoint from M (no endpoint of W has touched M yet).

The sign-reversing involution. On the disjoint union $\bigsqcup_{i=1}^{k} \mathcal{A}_i$ define Φ as follows.

1. (*Backtrack-first*) If $\tau_{\text{back}}(W) < \tau_{\text{touch}}(M, W)$, let uv be the edge of the first backtrack (so $W = \cdots uvu \cdots$ there). Set

$$\Phi(M, W) := \big(M \cup \{uv\},\ \text{erase}_{uv}(W)\big).$$

   Because uv is disjoint from M, $M \cup \{uv\}$ is a valid matching; the walk length drops from $2i$ to $2(i-1)$, so $\Phi(M, W) \in \mathcal{A}_{i-1}$ and the sign flips.

2. (*Touch-first*) If $\tau_{\text{touch}}(M, W) \leqslant \tau_{\text{back}}(W)$, let $uv \in M$ be the matching edge first touched by W (say $w_t = u$ at the touch time). Set

$$\Phi(M, W) := \big(M \setminus \{uv\},\ \text{splice}_{uv}(W)\big).$$

   This increases the walk length to $2(i+1)$ and keeps a matching, so $\Phi(M, W) \in \mathcal{A}_{i+1}$ and again the sign flips.

Define $\text{sign}(M, W) = (-1)^{|M|+k-1}$, and note that $\text{sign}(\Phi(M, W)) = -\text{sign}(M, W)$.

The only time Φ fails to produce another element of $\bigsqcup_{i=1}^{k} \mathcal{A}_i$ is in the backtrack-first case when $i = 1$ (erasing would create a length-0 walk, which lies outside the union).

In particular, it is possible to pair up elements in $\bigsqcup_{i=1}^{k} \mathcal{A}_i$ as $\{(M, W), (M', W')\}$ such that $\text{sign}(M, W) = -\text{sign}(M', W')$ with the exception of the case when W is a length-2 treelike walk disjoint from M.

Consequently, in the alternating signed sum all the pairs cancel, and the surviving terms are described by

$$\sum_{(M,W): W \text{ length-2}, M \text{ vertex disjoint from } W} (-1)^{|M|+k-1} = 2k \cdot m_k,$$

which gives the claimed identity. □

References

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[MSS15] Adam Marcus, Daniel Spielman, and Nikhil Srivastava. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Annals of Mathematics. Second Series*, 182(1):307–325, 2015. 1