CS 496 Lecture 13: Trickle-Down in High-Dimensional Expanders

Sidhanth Mohanty

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Some of the material here has been adapted from a note I wrote a couple years ago [Moh22].

1 Spectral independence and link expansion

Last lecture, we saw how to prove that the down-up walk has a spectral gap from link expansion. In this lecture, we will first recall the local-to-global theorem from last lecture and obtain the spectral independence theorem from two lectures ago as a corollary.

Recall the following two statements.

Theorem 1.1. *Given a d-dimensional simplicial complex X, define γ_j as $\min_{U \in j\text{-links}} \gamma(M_U)$. Then for $0 \leqslant k \leqslant d-1$:*

$$\gamma(P_k^{\Delta}) = \gamma(P_{k+1}^{\nabla}) \geqslant \frac{1}{k+2} \prod_{j=-1}^{k-1} \gamma_j.$$

Theorem 1.2 (Spectral independence $\Rightarrow$ rapid mixing)**.** *Let μ be a distribution on $\mathcal{F} \subseteq \binom{[n]}{k}$. If for every pinning T the influence matrix satisfies $\|\Psi^{\mu|T}\|_{\text{op}} \leqslant \min\left\{\kappa, \frac{d-|T|}{2}\right\}$, then the down-up walk has spectral gap at least $k^{-O(\kappa)}$, hence mixes in $k^{O(\kappa)} \log n$ time.*

We now prove Theorem 1.2 using Theorem 1.1.

Proof of Theorem 1.2. Recall that the influence matrix of a distribution μ is the $n \times n$ matrix:

$$\Psi^{\mu|T}[i,j] = \mathbf{Pr}_{S \sim \mu|T}[i \in S | j \in S] - \mathbf{Pr}_{S \sim \mu|T}[i \in S].$$

The random walk matrix P_U of the 1-skeleton of a link of T is given by:

$$P_U[i,j] = \frac{1}{k-|T|} \cdot \frac{\mathbf{Pr}_{S \sim \mu|T}[i,j \in S]}{\mathbf{Pr}_{S \sim \mu|T}[i \in S]}.$$

Observe that $\Psi^{\mu|T}$ can be obtained by taking $P_U^{\top}$ and subtracting a rank-1 component that is self-adjoint under π. Since eigenvalues interlace under rank-1 updates, the spectral radius of $\Psi^{\mu|T}$ is at least $\lambda_2(P_U)$. The desired statement then follows by plugging in Theorem 1.1. $\square$

2 Trickle-down theorem

In this lecture, we will cover the *trickle-down theorem*—a method to lower bound the spectral gaps of all links in a complex by proving a lower bound on the spectral gap of only the top-level links. This was first proved by [Żuk03] (see also [BdlHV08, Theorem 5.6.1] for the sharpest version of this statement, and the work of Oppenheim [Opp18] for the formulation used in the modern era).

Theorem 2.1. *Given a d-dimensional simplicial complex X, $k \leqslant d-2$, and a lower bound γ on the spectral gap of all k-links. Then for every $(k-1)$-link U, either $\gamma(M_U) = 0$ or $\gamma(M_U) \geqslant 2 - \frac{1}{\gamma}$.*

Proof. It suffices to prove the statement for $k = 0$. In particular, we assume that the link of every vertex has spectral gap at least γ and show that this implies that the graph underlying $\mathcal{S}$ either has spectral gap at least $2 - \frac{1}{\gamma}$ or is disconnected.

We do so via the following chain of inequalities:

$$\begin{aligned}
\langle f, L_0^{\Delta} f\rangle_{\pi_0} &= \mathbf{E}_{\{v,w\}\sim L_0^{\Delta}}\left[\frac{(f_v - f_w)^2}{2}\right] \\
&= \mathbf{E}_{u\sim\pi_0}\mathbf{E}_{\{v,w\}\sim M_u}\left[\frac{(f_v - f_w)^2}{2}\right] \\
&\geqslant \gamma\mathbf{E}_{u\sim\pi_0}\mathbf{E}_{v,w\sim\pi_u}\left[\frac{(f_v - f_w)^2}{2}\right] \\
&= \gamma\langle f, (\mathrm{Id} - (P_0^{\Delta})^2) f\rangle \qquad \text{(by time reversibility).}
\end{aligned}$$

Suppose L_0^{Δ} has spectral gap α, then the spectral gap of $\mathrm{Id} - (P_0^{\Delta})^2$ is $1 - (1-\alpha)^2 = 2\alpha - \alpha^2$, and consequently the above is at least:

$$\alpha\gamma(2-\alpha)\mathbf{E}_{v,w\sim\pi_0}\left[\frac{(f_v - f_w)^2}{2}\right].$$

Let f^* be a nonconstant vector that achieves the spectral gap of L_0^{Δ}. Then:

$$\alpha\mathbf{E}_{v,w\sim\pi_0}\left[\frac{(f_v^* - f_w^*)^2}{2}\right] \geqslant \alpha\gamma(2-\alpha)\mathbf{E}_{v,w\sim\pi_0}\left[\frac{(f_v^* - f_w^*)^2}{2}\right]$$

and consequently

$$\alpha \geqslant \alpha\gamma(2-\alpha).$$

Since we know $\alpha \geqslant 0$, to satisfy the above inequality either $\alpha = 0$ or $\alpha \geqslant 2 - \frac{1}{\gamma}$. □

Recall the notation $\gamma_j(X) := \min_{U \text{ } j\text{-link in } X} \gamma(M_U)$ for a simplicial complex X. As an immediate corollary of Theorem 2.1, we have:

Corollary 2.2. *Let X be a simplicial complex such that the 1-skeleton of every j-link is connected for $j \leqslant k-1$. Then:*

$$\gamma_j \geqslant 2 - \frac{1}{\gamma_{j+1}}.$$

Corollary 2.3. *The translation in the second eigenvalue world is $\lambda \to \frac{\lambda}{1-\lambda}$.*

Let X be a 2D complex. Observe that the trickle-down recurrence is nontrivial if $\gamma_{j+1} > \frac{1}{2}$. We cover some examples below of when the trickle-down theorem can be used to prove expansion in graphs.

Example 2.4 (Clique)**.** A single edge has a spectral gap of 2, and via trickle-down, the triangle has spectral gap $3/2$, the 4-clique has spectral gap $4/3$ and so on. In particular, one can use this to derive the optimal spectral gap of $\frac{n}{n-1}$ for the n-clique.

Example 2.5 (Complete multipartite complex)**.** Consider the simplicial complex where the vertices are split into $d+1$ parts $V_1, \dots, V_{d+1}$, and then one considers the d-dimensional complex obtained by taking all faces of size $d+1$ with exactly one element per V_i. Much like the clique, one can recursively trickle-down until the link looks like a complete bipartite graph, which has a spectral gap of 1, which implies that all complete multipartite graphs have spectral gap 1.

References

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[Moh22] Sidhanth Mohanty. *Local-to-global theorems for high-dimensional expansion*. 2022. Available at `https://sidhanthm.com/pdf/local-to-global.pdf`. 1

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[Żuk03] Andrzej Żuk. Property (T) and Kazhdan constants for discrete groups. *Geometric & Functional Analysis GAFA*, 13(3):643–670, 2003. 2