

CS 496 Lecture 14: Fast Mixing for Basis Exchange Walk

Sidhanth Mohanty

October 29, 2025

In today's lecture, we will cover the result of Anari, Liu, Oveis Gharan, and Vinzant [ALOGV19] that the down-up walk on *bases of a matroid* (e.g. the down-up walk on spanning trees of a graph) mixes rapidly.

1 The down-up walk on spanning trees has a spectral gap

Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices. Denote by $\mathcal{T}(G)$ the set of spanning trees of G . The *down-up walk* (also known as the *basis exchange walk*) on $\mathcal{T}(G)$ is the Markov chain that, from a current tree $T \in \mathcal{T}(G)$,

1. chooses an edge $e \in T$ uniformly at random and deletes it, creating two connected components of $T \setminus \{e\}$, and then
2. chooses uniformly at random an edge $f \in E \setminus T$ that reconnects those two components (equivalently, one for which $(T \setminus \{e\}) \cup \{f\}$ is a spanning tree), and moves to $T' := (T \setminus \{e\}) \cup \{f\}$.

The stationary distribution of this Markov chain with transition matrix P is the uniform distribution on $\mathcal{T}(G)$. We are interested in obtaining a handle on the spectral gap $\gamma(P) = 1 - \lambda_2(P)$.

The simplicial complex of spanning trees. Let X be the simplicial complex of the forests of G . In particular, consider the k -dimensional simplicial complex for $k = n - 2$ obtained by defining its k -faces as spanning trees of G , and obtaining lower dimensional faces via subset closure. Recall that the *link* of a face $U \in X$ is described by:

$$\text{Link}_X(U) = \{\sigma \setminus U : \sigma \in X, U \subseteq \sigma\}.$$

Recall that we used P_k^∇ to denote the transition matrix of the down-up walk on k -faces.

Recall the notation $\gamma(\text{Link}_X(U))$ to denote the spectral gap of the weighted random walk on the 1-skeleton of $\text{Link}_X(U)$, as well as

$$\gamma_j := \min_{U \in X(j)} \gamma(\text{Link}_X(U)) \quad (-1 \leq j \leq k - 2).$$

Two lectures ago, we proved the following local-to-global theorem of Alev and Lau [AL20].

Theorem 1.1 (Local-to-global). *We have:*

$$\gamma(P_k^\nabla) \geq \frac{1}{k+1} \prod_{j=-1}^{k-2} \gamma_j.$$

In the setting of spanning trees, we show that $\gamma_j \geq 1$ for all $-1 \leq j \leq k-2$, which immediately yields via [Theorem 1.1](#) that

$$\gamma(P_k^\nabla) \geq \frac{1}{k+1} = \frac{1}{n-1}.$$

To this end, we use the *trickle-down theorem*.

Lemma 1.2 (Trickle-down). *Suppose that for some $j \leq k-2$ every j -link has spectral gap at least $\gamma > 0$, and every $(j-1)$ -link is connected. Then every $(j-1)$ -link has spectral gap at least $2 - \frac{1}{\gamma}$.*

Our proof strategy to prove the desired lower bound on γ_j is to prove that $\gamma_{k-2} \geq 1$, and verify that all links are connected. The lower bound on γ_j is then an immediate consequence of [Lemma 1.2](#).

Structure of links. Fix a forest $U \in X$ and consider $\text{Link}_X(U)$. For two distinct components C, C' of U , set

$$E_{C,C'} := \{e \in E : e \text{ has one endpoint in } C \text{ and the other in } C'\}.$$

Then the 0-faces of $\text{Link}_X(U)$ are precisely the edges in $\bigsqcup_{C \neq C'} E_{C,C'}$, and two vertices e, e' of the link span an edge (i.e. can be added simultaneously to U while maintaining acyclicity) if and only if they lie in *different* parts $E_{C,C'}$. Thus the 1-skeleton of $\text{Link}_X(U)$ is a (possibly weighted) *complete multipartite graph* with parts $\{E_{C,C'}\}$. In particular, it is connected whenever there are at least two parts.

At dimension- $(k-2)$, the picture simplifies further. If $|U| = n-3$ then U has exactly three components A, B, C and $\text{Link}_X(U)$ is (the 1-skeleton of) a complete tripartite graph with parts $E_{A,B}$, $E_{A,C}$, and $E_{B,C}$ where all the edges have weight equal to 1.

A spectral lemma for complete multipartite links. We next analyze the eigenvalues of dimension- $(k-2)$ links.

Lemma 1.3. *For every $(k-2)$ -face U , the link-walk on $\text{Link}_X(U)$ has spectral gap at least 1.*

Proof. When $|U| = n-3$, the forest U has exactly three components A, B, C , and $\text{Link}_X(U)$ is a complete tripartite graph with parts $E_{A,B}$, $E_{A,C}$, $E_{B,C}$. Let e and f be edges in different parts. Since $U \cup \{e, f\}$ is a spanning tree, the weight of $\{e, f\}$ in the link is equal to 1. In particular, the link is the complete tripartite graph with all weights equal to 1. The unnormalized adjacency matrix is equal to $\mathbf{1}\mathbf{1}^\top - \mathbf{1}_{E_{A,B}}\mathbf{1}_{E_{A,B}}^\top - \mathbf{1}_{E_{B,C}}\mathbf{1}_{E_{B,C}}^\top - \mathbf{1}_{E_{A,C}}\mathbf{1}_{E_{A,C}}^\top$. Since this matrix is of the form rank-1 matrix – PSD matrix, it has at most one positive eigenvalue. This structure remains after normalizing too, so the random walk matrix of the link has at most 1 positive eigenvalue, which implies that the spectral gap must be at least 1. $\square$

Finally, in order to apply the trickle-down theorem ([Lemma 1.2](#)), we observe that the 1-skeletons of all links are connected.

Lemma 1.4. *For the complex on spanning trees of a connected graph G , every link is connected.*

Proof. Any link is a complete multipartite graph whose parts are the edge-sets $E_{C,C'}$ between distinct components C, C' of the forest τ . If there are at least two parts, such a graph is connected. $\square$

Combining [Theorem 1.1](#) with the above observations implies that the down-up walk mixes rapidly.

Theorem 1.5 (Spectral gap for the down-up walk on spanning trees). *Let G be a connected graph on n vertices, and let P be the down-up walk on $\mathcal{T}(G)$. Then*

$$\gamma(P) \geq \frac{1}{n-1}.$$

Equivalently, the (non-lazy) walk contracts $L^2(\mu)$ -deviations by a factor at most $1 - \frac{1}{n-1}$ in each step.

Remark 1.6 (Matroids and random bases). The argument extends to any *matroid*. As a special case distinct from the spanning tree example, consider the following setup. Let $v_1, \dots, v_m \in \mathbb{R}^n$ span $\mathbb{R}^n$. Consider the down-up walk on $k = n$ -sized subcollections that are bases and span $\mathbb{R}^n$. One can similarly partition the elements of a $(k-2)$ -link into equivalence classes and verify that the link is the complete multipartite graph on those equivalence classes and have spectral gap at least 1. Via the trickle-down and local-to-global theorems, one may establish a spectral gap of $1/n$ on the down-up walk in this instance.

References

- [AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd annual ACM SIGACT symposium on theory of computing*, pages 1198–1211, 2020. [1](#)
- [ALOGV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: high-dimensional walks and an fpras for counting bases of a matroid. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 1–12, 2019. [1](#)