CS 496 Lecture 17: Equiangular lines and multiplicity of λ_2

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1 Equiangular lines

Fix an angle $\theta \in [0, \pi]$ and write $\alpha := \cos\theta \in [-1, 1]$. A set $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ of lines in $\mathbb{R}^d$ is *equiangular with angle* θ if one can orient each line by a unit vector $v_i \in \ell_i$ so that

$$\langle v_i, v_j \rangle \in \{\pm\alpha\} \qquad \text{for all } i \neq j.$$

Let $N(d, \alpha)$ denote the largest n for which such a set exists in $\mathbb{R}^d$.

The punchline of today's lecture is that $N(d, \alpha)$ is controlled by an apparently unrelated parameter: the multiplicity of the second largest adjacency eigenvalue of a bounded-degree graph, which was proved by [JTY+21].

Theorem 1.1. *Let $\alpha \in (0, 1)$ and set*

$$\lambda = \frac{1-\alpha}{2\alpha}.$$

(i) Let k be the smallest integer such that there exists a connected graph H on k vertices with spectral radius $\rho(A_H) = \lambda$. Then:

$$N(d, \alpha) = (1 \pm o(1)) \frac{k}{k-1} d\,.$$

(ii) If λ is not the spectral radius of any finite connected graph, then

$$d \leqslant N(d, \alpha) \leqslant d + o(d).$$

1.1 From equiangular lines to Gram matrices and graphs

Given a configuration $\{v_1, \ldots, v_n\}$ with $\langle v_i, v_j \rangle \in \{\pm\alpha\}$, define a graph G on vertex set $[n]$ by declaring an edge $\{i, j\}$ exactly when $\langle v_i, v_j \rangle = -\alpha$. Let A_G be its adjacency matrix and J the all-ones matrix. Then the $n \times n$ *Gram matrix*

$$M = \big(\langle v_i, v_j \rangle\big)_{i,j=1}^n = (1-\alpha) I_n - \alpha(2A_G - J) = (1-\alpha) I_n - 2\alpha A_G + \alpha J. \tag{1}$$

Conversely, any symmetric positive semidefinite matrix of the form (1) with 1's on the diagonal is a Gram matrix of unit vectors with the desired inner products.

For later use, note the convenient parameterization

$$(1-\alpha) I_n - 2\alpha\, A_G = 2\alpha\,(\lambda I_n - A_G), \qquad \text{where} \quad \lambda = \frac{1-\alpha}{2\alpha}. \tag{2}$$

Proposition 1.2 (Graph-based construction)**.** *Let $\alpha \in (0,1)$ and $\lambda = (1-\alpha)/(2\alpha)$. Suppose there exists a connected graph H on k vertices with spectral radius $\rho(A_H) = \lambda$, and let k be minimum with this property. For any integer $m \geqslant 1$, consider the graph G that is the disjoint union of m copies of H (so $n = mk$). Then the matrix M in (1) is positive semidefinite and*

$$\operatorname{corank}(M) = m - 1, \qquad \operatorname{rank}(M) = m(k-1) + 1.$$

Consequently one obtains an equiangular set of $n = mk$ lines in $\mathbb{R}^d$ with

$$d = \operatorname{rank}(M) = m(k-1) + 1, \qquad \text{so} \qquad n = \frac{k}{k-1}\,d - \frac{k}{k-1}.$$

In particular $N(d,\alpha) \geqslant \frac{k}{k-1}\,d - O(1)$.

Proof. Observe that $(1-\alpha)I - 2\alpha A_G$ is PSD and has corank m. Adding a rank-1 matrix αJ decreases the corank by at most 1 — in particular, the rank is $m(k-1)+1$. Interpreting M as a Gram matrix yields the construction of equiangular lines. □

When λ is not the spectral radius of any graph adjacency matrix, we simply resort to the construction $M = (1-\alpha)I + \alpha J$, which is PSD.

We now focus on the upper bound, and assume the case where $\lambda = (1-\alpha)/(2\alpha)$ is *not* the spectral radius of any finite connected graph. The strategy is to show that every equiangular configuration induces a bounded-degree graph G whose adjacency matrix has the number λ as a second eigenvalue, and that the multiplicity of the second eigenvalue in bounded-degree graphs is $o(n)$.

From vectors to a bounded-degree graph. Let $\{v_i\}_{i=1}^n$ be unit vectors realizing n equiangular lines at angle θ in $\mathbb{R}^d$.

Claim 1.3 (Switching to bounded "negative degree")**.** There exists a choice of signs $\sigma_i \in \{\pm 1\}$ such that, writing $w_i = \sigma_i v_i$ and defining G by placing an edge $\{i,j\}$ iff $\langle w_i, w_j\rangle = -\alpha$, the maximum degree satisfies

$$\Delta(G) \;\leqslant\; \Delta_\alpha,$$

for some constant Δ_α depending only on α (not on n or d).

With this switching in place, our Gram matrix is still $M = (1-\alpha)I - 2\alpha A_G + \alpha J \succeq 0$. By PSDness of M and Cauchy's interlacing theorem, A_G has at most one eigenvalue strictly larger than λ.

It follows that every other appearance of λ in the spectrum must be as a *second* eigenvalue. Thus, we have:

$$n - d \;\leqslant\; \operatorname{corank}(M) \;\leqslant\; 1 + \operatorname{mult}_G(\lambda_2), \tag{3}$$

where λ_2 denotes the (global) second-largest eigenvalue of A_G (counted with multiplicity). Thus to prove $n \leqslant d + o(d)$ it suffices to show:

Theorem 1.4. *For every fixed Δ and every n-vertex connected graph G with $\Delta(G) \leqslant \Delta$,*

$$\operatorname{mult}_G(\lambda_2) = o(n).$$

It remains to prove Theorem 1.4.

Heavy vertices and their clustering. Let $B_r(v)$ denote the radius-r neighborhood around a vertex v, and write $\rho(X)$ for the spectral radius of the adjacency matrix of a graph X.

Definition 1.5 (Heavy vertices)**.** A vertex v is *r-heavy* if $\rho\big(G[B_r(v)]\big) > \lambda_2(G)$.

Lemma 1.6 (Heavy vertices are clustered)**.** *If u and v are r-heavy and $\operatorname{dist}(u,v) \geqslant 2r+1$, then A_G has at least two eigenvalues strictly larger than $\lambda_2(G)$, a contradiction. Hence all r-heavy vertices lie inside a single ball of radius $2r$, and therefore there are at most Δ^{2r} heavy vertices.*

Proof. When $\operatorname{dist}(u,v) \geqslant 2r+1$, the induced balls $B_r(u)$ and $B_r(v)$ are vertex-disjoint. Let x_u (resp. x_v) be unit Perron eigenvectors of $A_{G[B_r(u)]}$ (resp. $A_{G[B_r(v)]}$). Then x_u and x_v have disjoint supports and Rayleigh quotients $> \lambda_2(G)$. By the variational characterization of eigenvalues, A_G has two eigenvalues $> \lambda_2(G)$. The bound on the number of heavy vertices is immediate from the degree bound of Δ. □

Delete all r-heavy vertices; by Lemma 1.6 this removes at most $\Delta^{2r} = o(n)$ vertices if $r = c\log n$ with $c > 0$ small. In the remaining graph G°, every r-ball has spectral radius at most $\lambda_2(G)$.

Choose $r' = c'\log\log n$. Let $S \subseteq V(G^\circ)$ be a *maximal* r'-separated set, i.e., any pair of vertices in S are at least r' apart, and every vertex of G° lies within distance $\leqslant r'$ of S. Such an S exists with

$$|S| \leqslant \frac{n}{\min_{v\in V(G^\circ)} |B_{r'}(v)|} \leqslant \frac{n}{\Theta(r')} = o(n),$$

We will use the notation $H := G^\circ \setminus S$.

Lemma 1.7. *Let $L := 2r'$. For every $v \in V(H)$,*

$$\big(A_H^L\big)_{vv} \;\leqslant\; \big(A_{G^\circ}^L\big)_{vv} - 1.$$

Consequently, for each v the spectral radius of the adjacency matrix of the ball $H[B_r(v)]$ satisfies

$$\rho\big(H[B_r(v)]\big)^L \;\leqslant\; \rho\big(G^\circ[B_r(v)]\big)^L - 1 \;\leqslant\; \lambda_2(G)^L - 1. \tag{4}$$

Proof. For a vertex $v \in V(H)$, let $s \in S$ be some vertex at distance-$\leqslant r'$. A length-$\leqslant r'$ path between v and s concatenated with its reverse forms a closed walk of length exactly $2t \leqslant 2r'$ through s. All such walks vanish upon deleting S, so the diagonal entry of $A^{2r'}$ drops by at least 1, which completes the proof. □

Trace method and multiplicity bound. For even t, $\operatorname{trace}(A_H^t) = \sum_v (A_H^t)_{vv}$ counts closed t-walks. Since any closed t-walk from v stays within $B_t(v)$,

$$(A_H^t)_{vv} = \big(A^t_{H[B_t(v)]}\big)_{vv} \;\leqslant\; \rho\big(H[B_t(v)]\big)^t.$$

Apply this with $t = r$ and use (4):

$$\operatorname{trace}(A_H^r) \;\leqslant\; \sum_{v\in V(H)} \rho\big(H[B_r(v)]\big)^r \;\leqslant\; |V(H)| \cdot \big(\lambda_2(G)^L - 1\big)^{r/L}.$$

On the other hand, by Cauchy interlacing, deleting $o(n)$ vertices reduces the multiplicity of λ_2 by at most $o(n)$: if $m = \mathrm{mult}_G(\lambda_2)$ and $m_H = \mathrm{mult}_H(\lambda_2)$, then $m_H \geqslant m - o(n)$. Therefore,

$$m_H\,\lambda_2(G)^r \;\leqslant\; \mathrm{trace}(A_H^r) \;\leqslant\; |V(H)| \cdot \big(\lambda_2(G)^L - 1\big)^{r/L}.$$

Rearranging,

$$m_H \;\leqslant\; |V(H)| \cdot \Big(1 - \lambda_2(G)^{-L}\Big)^{r/L}. \tag{5}$$

Now take $r = c\log n$ and $L = 2r' = \Theta(\log\log n)$. Since $\lambda_2(G) \geqslant 1$ and $L \to \infty$, the factor $\big(1 - \lambda_2(G)^{-L}\big)^{r/L}$ tends to 0 faster than any $1/\mathrm{polylog}(n)$. Hence $m_H = o(n)$ and therefore $m = m_H + o(n) = o(n)$, proving Theorem 1.4.

Finally, inserting $\mathrm{mult}_G(\lambda_2) = o(n)$ into (3) yields

$$n \;\leqslant\; d + 1 + o(n), \qquad \text{hence} \qquad n \;\leqslant\; d + o(d),$$

which completes the proof of Theorem 1.1(ii).

References

[JTY+21] Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang, and Yufei Zhao. Equiangular lines with a fixed angle. *Annals of Mathematics*, 194(3):729–743, 2021. 1