CS 496 Lecture 18: Girth-density tradeoffs I: Irregular Moore bound

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November 19, 2025

1 Girth–density tradeoffs in graphs and hypergraphs

In this lecture we study how large the girth of a graph or hypergraph can be as a function of its density. Throughout, for a graph $G = (V, E)$ we let $n = |V|$, $m = |E|$, and write

$$d := \frac{2m}{n}$$

for the average degree. The *girth* $g(G)$ is the length of the shortest cycle in G. For a k-uniform hypergraph we will use the notion of a cycle called an *even cover*.

We will see three types of Moore-type bounds:

- For regular graphs: a d-regular n-vertex graph must contain a cycle of length at most about $2 \log_{d-1} n$.
- For irregular graphs (Alon–Hoory–Linial): for an n-vertex graph of average degree $d > 2$ we still have $g(G) \lesssim 2 \log_{d-1} n$.
- For k-uniform hypergraphs: if H has "many" hyperedges, then it contains a short even cover; quantitatively, when H has roughly

$$m \gtrsim n\left(\frac{n}{r}\right)^{\frac{k}{2}-1} \log n$$

  hyperedges, it has an even cover of size $O(r \log n)$.

All of these will be proved (or at least sketched) using *spectral double counting*, a technique pioneered by Guruswami, Kothari, and Manohar [GKM22].

The classical Moore bound for regular graphs

Theorem 1.1 (Moore bound for d-regular graphs)**.** *Let G be a d-regular graph on n vertices with $d \geqslant 3$ and girth $g(G)$. Then*

$$g(G) \leqslant 2 \log_{d-1} n + 2\,.$$

Proof. Fix a vertex v and suppose $g(G) > 2R+1$. Then the ball of graph radius R around v is a tree. In a d-regular tree, the number of vertices at distance at most R from the root is

$$1 + d\sum_{i=0}^{R-1}(d-1)^i = 1 + d\,\frac{(d-1)^R - 1}{(d-1)-1} \geqslant (d-1)^R.$$

Hence $n \geqslant (d-1)^R$, so $R \leqslant \log_{d-1} n$. Taking $R = \lfloor (g(G)-1)/2 \rfloor$ and rearranging gives the stated bound on $g(G)$. $\square$

1.1 Irregular Moore bound via the Bethe Hessian

We now show how to prove an *irregular* Moore bound (for graphs of arbitrary degrees) via spectral double counting.

Theorem 1.2 (Irregular Moore bound)**.** *Let G be a graph on n vertices with average degree $d > 2$ and girth $g(G)$. Then*

$$g(G) \leqslant 2\log_{d-1} n + 2\,.$$

The reason the above statement is remarkable is that a tree, which has no cycles, has average degree $2 - 2/n$, but if one adds εn edges, then the above statement implies the existence of a length-$O(\log n)$ cycle.

We now dive into the proof of the irregular Moore bound. We start by introducing relevant background on *nonbacktracking walks*.

Nonbacktracking walks. Let G be an arbitrary graph with adjacency matrix A and degree matrix $D = \operatorname{diag}(\deg(v_1), \ldots, \deg(v_n))$. A walk $(v_0, v_1, \ldots, v_\ell)$ is *nonbacktracking* if $v_{i+1} \neq v_{i-1}$ for all $1 \leqslant i \leqslant \ell - 1$.

For each integer $\ell \geqslant 0$ we define an $n \times n$ matrix $A^{(\ell)}$ by

$$\left(A^{(\ell)}\right)_{ij} = \#\{\text{nonbacktracking walks of length } \ell \text{ from } i \text{ to } j\}.$$

Thus $A^{(0)} = I$ and $A^{(1)} = A$. One may prove that for all $\ell \geqslant 2$:

$$A^{(\ell)} = A \cdot A^{(\ell-1)} - (D-I)\cdot A^{(\ell-2)}. \tag{1}$$

The Bethe Hessian and its inverse series. Define the *Bethe Hessian* of G as the t-parametrised matrix

$$H(t) \;:=\; (D-I)t^2 - At + I.$$

We will show that the inverse of $H(t)$, when it exists as a power series around $t = 0$, is

$$H(t)^{-1} \;=\; \frac{I + At + A^{(2)}t^2 + A^{(3)}t^3 + \cdots}{1 - t^2}.$$

Set

$$F(t) \;:=\; \sum_{\ell=0}^{\infty} A^{(\ell)} t^\ell \;=\; I + At + A^{(2)}t^2 + \cdots\,.$$

Lemma 1.3 (Inverse identity for the Bethe Hessian). *Formally (as a power series in t) we have*

$$H(t)\,F(t) \;=\; (1-t^2)I.$$

Consequently, whenever the series defining $F(t)$ converges and $1-t^2 \neq 0$,

$$H(t)^{-1} \;=\; \frac{F(t)}{1-t^2} \;=\; \frac{I + At + A^{(2)}t^2 + A^{(3)}t^3 + \cdots}{1-t^2}.$$

Proof. Write $H(t) = I - At + (D-I)t^2$, so

$$H(t)F(t) \;=\; \big(I - At + (D-I)t^2\big)\sum_{\ell \geqslant 0} A^{(\ell)}t^\ell.$$

We compare coefficients of t^m on both sides.

Coefficient of t^0. Only the I term contributes: we get $A^{(0)} = I$.

Coefficient of t^1. We have $A^{(1)} - AA^{(0)} = A - A = 0$.

Coefficient of t^2. We obtain

$$A^{(2)} - AA^{(1)} + (D-I)A^{(0)} \;=\; A^{(2)} - A^2 + (D-I).$$

But one checks directly that $A^{(2)} = A^2 - D$ (each length-2 walk is either non-backtracking, contributing to $A^{(2)}$, or backtracks, contributing to D), so the coefficient simplifies to $-I$.

Thus so far we have $I - t^2 I$.

Coefficient of t^m for $m \geqslant 3$. For $m \geqslant 2$ the coefficient is

$$A^{(m)} - AA^{(m-1)} + (D-I)A^{(m-2)}.$$

Using the recurrence (1) with $\ell = m-1 \geqslant 2$,

$$AA^{(m-1)} \;=\; A^{(m)} + (D-I)A^{(m-2)}.$$

Substituting this into the coefficient gives zero. Therefore

$$H(t)F(t) \;=\; I - t^2 I \;=\; (1-t^2)I$$

as formal power series, which proves the claim. □

Radius of convergence and positivity. Let

$$F(t) = \sum_{\ell \geqslant 0} A^{(\ell)}t^\ell$$

be viewed as a matrix-valued power series, and let R denote its radius of convergence (with respect to any operator norm). By Lemma 1.3, on the disk $|t| < R$ we have an analytic inverse

$$H(t)^{-1} \;=\; \frac{F(t)}{1-t^2}.$$

In particular $H(t)$ is invertible for all $|t| < R$.

Because $H(0) = I$ is strictly positive definite and $H(t)$ is real symmetric for real t, the following easy fact holds: if a symmetric matrix-valued function $M(t)$ is positive definite at $t = 0$ and invertible for all t in an interval $(-\rho, \rho)$, then $M(t)$ is positive definite for all $t \in (-\rho, \rho)$. Indeed, every eigenvalue is a continuous real-valued function of t, and no eigenvalue can cross 0 without violating invertibility.

Applied to $H(t)$, this shows that for all $|t| < R$ the Bethe Hessian $H(t)$ is positive definite.

Lemma 1.4. *Let G be any graph on n vertices with average degree $d > 1$. Then for $t = 1/(d-1)$ we have*

$$\mathbf{1}^\top H(t)\,\mathbf{1} \;=\; 0,$$

where $\mathbf{1}$ is the all-ones vector. In particular, $H(t)$ is not positive definite for $t = 1/(d-1)$ and thus $R \leqslant 1/(d-1)$.

Proof. Compute

$$\mathbf{1}^\top H(t)\,\mathbf{1} \;=\; t^2\,\mathbf{1}^\top (D-I)\mathbf{1} \;-\; t\,\mathbf{1}^\top A\mathbf{1} \;+\; \mathbf{1}^\top\mathbf{1}.$$

Now $\mathbf{1}^\top\mathbf{1} = n$ and

$$\mathbf{1}^\top A\mathbf{1} = \sum_{i,j} A_{ij} = 2m = dn,$$

while

$$\mathbf{1}^\top (D-I)\mathbf{1} = \sum_{v} (\deg(v) - 1) = 2m - n = n(d-1).$$

Hence

$$\mathbf{1}^\top H(t)\,\mathbf{1} = n\big((d-1)t^2 - dt + 1\big).$$

The quadratic

$$f(t) := (d-1)t^2 - dt + 1$$

vanishes at $t = 1/(d-1)$, so $\mathbf{1}^\top H(1/(d-1))\mathbf{1} = 0$. If $H(t)$ were positive definite at that value of t, then $\mathbf{1}^\top H(t)\mathbf{1} > 0$, a contradiction. Thus $H(t)$ fails to be positive definite when $t = 1/(d-1)$.

As argued above, $H(t)$ is positive definite on $(-R, R)$. Therefore the interval $(-R, R)$ cannot contain $t = 1/(d-1)$, and hence $R \leqslant 1/(d-1)$. □

Bounding the radius from below using the girth. We now lower bound the radius of convergence R under the hypothesis that G has no short cycles.

Lemma 1.5. *Suppose G has girth $g(G) > 2\ell$.*

1. *For every pair of vertices i, j, there is at most one non-backtracking walk of length ℓ from i to j. Equivalently, all entries of $A^{(\ell)}$ lie in $\{0, 1\}$.*
2. *In particular, the spectral norm satisfies*

$$\|A^{(\ell)}\|_2 \leqslant \rho(A^{(\ell)}) \leqslant n,$$

since every row sum is at most n.

3. *For every integer k that is a multiple of ℓ, say $k = r\ell$, one has the entrywise inequality*

$$A^{(k)} \preceq \left(A^{(\ell)}\right)^r,$$

where $\preceq$ denotes entrywise comparison of nonnegative matrices. Consequently

$$\rho\left(A^{(k)}\right) \leqslant \rho\left(A^{(\ell)}\right)^r \leqslant n^{k/\ell}.$$

Proof. **Proof of Item 1.** Suppose there are two distinct non-backtracking walks of length ℓ between i and j. Their union contains a cycle; by tracing along the two walks until they diverge and then meet again, we obtain a simple cycle of length at most 2ℓ, contradicting $g(G) > 2\ell$.

Proof of Item 2. With entries in $\{0, 1\}$ the maximum row sum is at most n, so the spectral radius and the operator norm are at most n.

Proof of Item 3. Fix i, j and let $k = r\ell$. Any non-backtracking walk $(v_0, \ldots, v_k)$ of length k can be partitioned into the r blocks of length ℓ:

$$(v_0, \ldots, v_\ell), (v_\ell, \ldots, v_{2\ell}), \ldots, (v_{(r-1)\ell}, \ldots, v_{r\ell}).$$

By part (1), for each ordered pair of vertices (u, v) there is at most one non-backtracking walk of length ℓ from u to v. Therefore the number of non-backtracking k-walks from i to j is at most the number of length-r walks in the directed graph on the same vertex set whose adjacency matrix is $A^{(\ell)}$, namely $\left(A^{(\ell)}\right)^r_{ij}$. This yields the claimed entrywise inequality, and the spectral radius inequality follows because $\rho(M^r) = \rho(M)^r$, and the spectral radius is monotone on nonnegative matrices under entrywise domination. $\square$

The usual formula for the radius of convergence of a matrix power series $F(t) = \sum_{k \geqslant 0} B_k t^k$ with respect to any submultiplicative norm is

$$\frac{1}{R} = \limsup_{k \to \infty} \|B_k\|^{1/k}.$$

Applying this to $F(t)$ with $B_k = A^{(k)}$ and using Lemma 1.5(3) for multiples $k = r\ell$ gives

$$\frac{1}{R} \leqslant \limsup_{r \to \infty} \left\| A^{(r\ell)} \right\|_2^{1/(r\ell)} \leqslant \limsup_{r \to \infty} n^{1/\ell} = n^{1/\ell}.$$

Hence

$$R \geqslant n^{-1/\ell}. \tag{2}$$

We are now ready to prove the irregular Moore bound (Theorem 1.2).

Proof. Let $\ell = \lfloor g(G)/2 \rfloor - 1$. Then $g(G) > 2\ell$, so Lemma 1.5 applies and yields the lower bound (2): $R \geqslant n^{-1/\ell}$. On the other hand, Lemma 1.4 gives $R \leqslant 1/(d-1)$. Therefore

$$n^{-1/\ell} \leqslant \frac{1}{d-1},$$

which rearranges to

$$\ell \leqslant \log_{d-1} n.$$

Since $g(G) \leqslant 2\ell + 2$, we obtain $g(G) \leqslant 2 \log_{d-1} n + 2$. $\square$

References

[GKM22] Venkatesan Guruswami, Pravesh K Kothari, and Peter Manohar. Algorithms and certificates for Boolean CSP refutation: smoothed is no harder than random. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 678–689, 2022. 1