CS 496 Lecture 19: Girth-density tradeoffs II: Hypergraph Moore bound

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1 Hypergraphs, even covers, and the hypergraph Moore bound

Definition 1.1 (Even covers and girth of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a k-uniform hypergraph (each $C \in \mathcal{E}$ has $|C| = k$). A (multi)set $\mathcal{F} \subseteq \mathcal{E}$ of hyperedges is called an *even cover* if every vertex $v \in V$ lies in an even number of edges of $\mathcal{F}$. The *girth* of H is the minimum size $|\mathcal{F}|$ of a nonempty even cover in H.

Remark 1.2. Equivalently, if we view each hyperedge C as its incidence vector in $\{0,1\}^V$ and sum vectors modulo 2, an even cover is exactly a nonzero $\mathbb{F}_2$-linear dependence among the edge vectors.

We will prove the existence of short even covers using a graph associated to $\mathcal{H}$ called the *Kikuchi graph*, and then apply spectral double counting on this graph. The hypergraph Moore bound was first proved by Guruswami, Kothari, and Manohar [GKM22]. We follow the treatment of Hsieh, Kothari, and Mohanty [HKM23], who simplified the proof.

Theorem 1.3 (Hypergraph Moore bound)**.** *Let $k \geqslant 2$ and let r satisfy $k \leqslant r \leqslant O(n)$. There is an absolute constant C_k such that the following holds for every k-uniform hypergraph H on n vertices: if H has at least*

$$m \geqslant C_k n \left(\frac{n}{r}\right)^{\frac{k}{2}-1} \log n$$

hyperedges, then H contains an even cover of size $O_k(r \log n)$.

Definition 1.4 (Kikuchi graph, even k)**.** Let H be a k-uniform hypergraph on vertex set $[n]$, where k is even, and fix an integer r with $k \leqslant r \leqslant n$. The *Kikuchi graph* $K_r(H)$ has vertex set

$$V(K_r) \;=\; \binom{[n]}{r},$$

the family of all r-subsets of $[n]$. Two vertices $S, T \in \binom{[n]}{r}$ are adjacent if their symmetric difference $S \oplus T$ is a hyperedge of H. In that case we write

$$S \overset{C}{\longleftrightarrow} T \quad \text{with} \quad C := S \oplus T \in \mathcal{E}(H),$$

and think of the edge (S, T) as *colored* by C. The adjacency matrix of $K_r(H)$ is called the *Kikuchi matrix*.

The crucial connection between even covers and the Kikuchi graph is the following simple observation.

Observation 1.5 (Closed walks and even covers)**.** Let H be a k-uniform hypergraph with even k, and fix r. Consider a closed walk

$$S_1 \xrightarrow{C_1} S_2 \xrightarrow{C_2} \dots \xrightarrow{C_\ell} S_{\ell+1} = S_1$$

of length ℓ in $K_r(H)$, where each $C_i \in \mathcal{E}(H)$ and $S_i \oplus S_{i+1} = C_i$. Then

$$C_1 \oplus C_2 \oplus \cdots \oplus C_\ell = \varnothing.$$

In particular, the multiset $\{C_1, \dots, C_\ell\}$ is an even cover of H. Conversely, if H has no even cover of size at most ℓ, then in every such closed walk each hyperedge $C \in \mathcal{E}(H)$ appears an even number of times among $\{C_1, \dots, C_\ell\}$; we call such walks *trivial*.

Proof. From $S_i \oplus S_{i+1} = C_i$ and the fact that each S_i appears in exactly two of these equations, summing all of them modulo 2 yields

$$0 = S_1 \oplus S_1 \oplus \cdots \oplus S_\ell \oplus S_\ell = C_1 \oplus \cdots \oplus C_\ell.$$

Thus $\{C_1, \dots, C_\ell\}$ is an even cover. The converse statement follows by repeatedly cancelling hyperedges that occur an even number of times: if some hyperedge occurs an odd number of times, the residual set of hyperedges forms an even cover. □

Observation 1.5 shows that short nontrivial closed walks in the Kikuchi graph correspond to short even covers in H. Thus to show that H must contain a short even cover, it suffices to force the Kikuchi graph to have many short cycles, and then bound the number of trivial cycles.

Density of the Kikuchi graph. Let H be a k-uniform hypergraph on n vertices with m hyperedges, and consider its Kikuchi graph $K_r(H) = (V, E)$. For each hyperedge $C \in \mathcal{E}(H)$, to create an edge (S, T) of $K_r(H)$ coloured by C we must choose $k/2$ vertices from C and $r - k/2$ vertices from $[n] \setminus C$ to form S; the set T is then uniquely determined as $S \oplus C$. This yields

$$\binom{k}{k/2}\binom{n-k}{r-k/2}$$

ordered pairs (S, T), or half as many undirected edges. Summing over all m hyperedges we get

$$|E| \;=\; \frac{1}{2}\binom{k}{k/2}\binom{n-k}{r-k/2} m.$$

The number of vertices of $K_r(H)$ is $N := \binom{n}{r}$, so the average degree d of $K_r(H)$ satisfies

$$d = \frac{2|E|}{|V|} = \frac{\binom{k}{k/2}\binom{n-k}{r-k/2}}{\binom{n}{r}} \cdot m\,, \tag{1}$$

which implies the lower bound

$$d \geqslant \frac{1}{2}\left(\frac{r}{n}\right)^{k/2} m. \tag{2}$$

Spectral double counting on the Kikuchi graph. Let A denote the adjacency matrix of $K_r(H)$, let D be its degree matrix, and let d be its average degree. Introduce the diagonal matrix

$$\Gamma := D + dI$$

and the reweighted adjacency

$$A^{\mathrm{rw}} := \Gamma^{-1/2} A \Gamma^{-1/2}.$$

We will prove:

Lemma 1.6 (Reweighted Kikuchi bound)**.** *Let H be a k-uniform hypergraph on n vertices with even k, and let $K_r(H)$ be its Kikuchi graph with average degree d. Fix an even integer ℓ and suppose H has no even cover of size at most ℓ. Then*

$$\|A^{\mathrm{rw}}\|_{\mathrm{op}} < 2N^{1/\ell}\sqrt{\frac{\ell}{d}}.$$

Proof. Our proof proceeds via the trace power method. For simplicity, we assume that the graph is regular, and so every edge has weight $\frac{1}{d}$ (see [HKM23, Lemma 2.6] for the general case).

If H has no even cover of size-$\leqslant \ell$ then, by Observation 1.5, every closed length-ℓ walk in K_r uses each hyperedge an even number of times. Thus, we can encode a length-ℓ closed walk by:

- its starting vertex $S_1 \in V(K_r)$,
- a bit $b_i \in \{0, 1\}$ at each step i indicating whether we traverse a *new* hyperedge—one that is being used in the walk for the first time—or an *old* hyperedge—one that has been used earlier in the walk.
- a number in $[d]$ for every new hyperedge, specifying which of the d neighbors in the Kikuchi graph is walked to,
- a number in $[\ell]$ for every old hyperedge, specifying the timestamp of the first visit to that hyperedge.

The total number of valid encodings is $N \cdot 2^\ell \cdot d^{\ell/2} \cdot \ell^{\ell/2}$, and the weight of the walk is $d^{-\ell}$.

This results in an operator norm bound of:

$$\|A^{\mathrm{rw}}\|_{\mathrm{op}} \leqslant 2N^{1/\ell}\sqrt{\frac{\ell}{d}}.$$ □

From the operator inequality

$$A \preceq \lambda\Gamma \quad \text{with} \quad \lambda = \|\Gamma^{-1/2} A \Gamma^{-1/2}\|,$$

we obtain

$$\mathbf{1}^\top A \mathbf{1} < \lambda \mathbf{1}^\top \Gamma \mathbf{1} = \lambda \sum_{S \in V} (\deg(S) + d) = 2\lambda N d.$$

But $\mathbf{1}^\top A \mathbf{1} = 2|E| = Nd$, so if there is no even cover of size at most ℓ we deduce

$$d < 4\lambda d \leqslant 8N^{1/\ell}\sqrt{\ell d}$$

and hence

$$d < O\left(\ell N^{2/\ell}\right).$$

On the other hand, we showeed in (2) that

$$d \geqslant \frac{1}{2}\left(\frac{r}{n}\right)^{k/2} m.$$

Combining these inequalities and plugging in $\ell = r \log n$ yields an upper bound on m under the assumption that there is no even cover of size at most ℓ that is exactly the hypergraph Moore bound.

References

[GKM22] Venkatesan Guruswami, Pravesh K Kothari, and Peter Manohar. Algorithms and certificates for Boolean CSP refutation: smoothed is no harder than random. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 678–689, 2022. 1

[HKM23] Jun-Ting Hsieh, Pravesh K Kothari, and Sidhanth Mohanty. A simple and sharper proof of the hypergraph Moore bound. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2324–2344. SIAM, 2023. 1, 3