CS 496 Lecture 2: The Second Eigenvalue

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1 Cheeger wrapup and spectral expansion

Last time, we ended the class with a proof sketch of the "easier" direction of the Cheeger inequalities. We recall the Cheeger inequalities from before, and flesh out the details.

Theorem 1.1 (Cheeger inequalities, [AM85]). *The following inequality relating the second eigenvalue and edge expansion is true:*

$$2\Phi_E(G) \geqslant 1 - \lambda_2(P_G) \geqslant \frac{\Phi_E(G)^2}{2}\,.$$

Proof that small second eigenvalue gives strong edge expansion. Observe that for any set S such that $\pi(S) \leqslant \frac{1}{2}$, let f_S be the test function given by $\mathbf{1}_S - \pi(S)\cdot\mathbf{1}$. Since $\langle f_S, \mathbf{1}\rangle_\pi = 0$, we have:

$$\begin{aligned}
1 - \lambda_2(P_G) &\leqslant \frac{\langle f_S, (I-P_G)f_S\rangle_\pi}{\|f_S\|_\pi^2} \\
&= \frac{\mathbf{Pr}_{u\sim\pi,\, v\sim_G u}[u\in S, v\notin S] + \mathbf{Pr}_{u\sim\pi,\, v\sim_G u}[u\notin S, v\in S]}{2\pi(S)(1-\pi(S))} \\
&= \frac{2\mathbf{Pr}_{u\sim\pi,\, v\sim_G u}[u\in S, v\notin S]}{\pi(S)} \leqslant 2\Phi_E(G)\,.
\end{aligned}$$

Let us justify the equality going from the first to the second line. We will establish a more general identity. For any functions f, g,

$$\langle f, (I-P)g\rangle_\pi = \mathbf{E}_{u\sim\pi}\frac{1}{2}\mathbf{E}_{v\sim_G u}(f(v)-f(u))\cdot(g(v)-g(u))\,.$$

One obtains a handle on the numerator by plugging in $f = g = f_S$ into the above equality. One can see this by the following chain of equalities:

$$\begin{aligned}
\langle f, (I-P_G)g\rangle_\pi &= f^\top \operatorname{diag}(\pi)D^{-1}(D-A_G)g \\
&= f^\top \operatorname{diag}(\pi)D^{-1}\sum_{uv\in E(G)}(\mathbf{1}_u-\mathbf{1}_v)(\mathbf{1}_u-\mathbf{1}_v)^\top g \\
&= f^\top \frac{I}{2m}\sum_{uv\in E(G)}(\mathbf{1}_u-\mathbf{1}_v)(\mathbf{1}_u-\mathbf{1}_v)^\top g \\
&= \frac{1}{2m}\sum_{uv}(f(u)-f(v))(g(u)-g(v))
\end{aligned}$$

$$= \frac{1}{2}\mathbf{E}_{uv\sim E(G)}(f(u)-f(v))(g(u)-g(v))$$
$$= \frac{1}{2}\mathbf{E}_{u\sim\pi}\mathbf{E}_{v\sim_G u}(f(u)-f(v))(g(u)-g(v))\,.$$

For the denominator, observe that for any zero-mean function f, we have $\|f\|_\pi^2 = \mathbf{Var}_\pi[f] = \frac{1}{2}\mathbf{E}_{u,v\sim\pi}(f(u)-f(v))^2$. Plugging in $f = f_S$ reveals that the denominator is equal to $\pi(S)(1-\pi(S))$. □

The takeaway from the above is that expansion defined directly in terms of eigenvalues merits understanding.

Definition 1.2 (Spectral expansion)**.** We define the *spectral expansion* $\lambda(G)$ of a graph G as $\lambda_2(P_G)$. We say a graph G is a *λ-spectral expander* if $\lambda(G) \leqslant \lambda$, and we say that G is a *λ-two-sided spectral expander* if $\lambda(G) \leqslant \lambda$, and $|\lambda_n(P_G)| \leqslant \lambda$.

On existence of spectral expanders. Before we go ahead and develop a theory for spectral expanders, it is natural to wonder what some examples of spectral expanders with small λ are—after all, the only example we have seen thus far is the clique.

It turns out that random graphs are excellent spectral expanders; indeed they are as good spectral expanders as one may hope for.

Theorem 1.3 ([Nil91, Fri08])**.** *Let G be an n-vertex d-regular graph for $d \geqslant 3$. Then $\lambda(G) \geqslant \frac{2\sqrt{d-1}-o_n(1)}{d}$. Suppose G is a random d-regular n-vertex graph. Then, with high probability G is a $\frac{2\sqrt{d-1}+o_n(1)}{d}$-two-sided spectral expander.*

The proof of the second part of this statement is quite involved and beyond the scope of this class, but perhaps we will see some of the ideas that go into proving this in a future lecture!

We give a bit of a historical aside on the above theorem. The conjecture was made by Alon [Alo86] in the 80s, and proved by Friedman [Fri08] only 20 years later, but his proof was quite involved and spanned 100+ pages. A simpler proof was given by Bordenave [Bor19] (30 pages), and very recently, an even simpler proof using totally different ideas was given by Chen, Garza Vargas, Tropp, and van Handel [CGVTvH24].

In a recent breakthrough from a few months ago, Huang, McKenzie, and Yau [HMY24] proved that G is a *Ramanujan graph*, i.e., a $\frac{2\sqrt{d-1}}{d}$-two-sided spectral expander (without the $o(1)$) with probability ≈ 0.69.

2 Properties of λ-spectral expanders

The spectral expansion of a graph also controls the density of its subgraphs via the *expander mixing lemma*. In a strong spectral expander, the density of subgraphs looks like that in a random graph.

Lemma 2.1 (Expander Mixing Lemma)**.** *In any λ-two-sided spectral expander G, and $S, T \subseteq V(G)$:*

$$\mathbf{Pr}_{u\sim\pi,v\sim_G u}[u\in S, v\in T] = \pi(S)\pi(T) \pm \lambda\sqrt{\pi(S)\cdot\pi(T)}\,.$$

Proof. Observe that we can write:

$$\begin{aligned}\mathbf{Pr}_{u\sim\pi, v\sim_G u}[u \in S, v \in T] &= \langle \mathbf{1}_S, P_G \mathbf{1}_T \rangle_\pi \\ &= \langle \mathbf{1}_S, \mathbf{1} \cdot \pi(T) \rangle_\pi + \langle \mathbf{1}_S, P(\mathbf{1}_T - \pi(T) \cdot \mathbf{1}) \rangle_\pi \\ &= \pi(S) \cdot \pi(T) \pm \lambda \sqrt{\pi(S)\pi(T)}\,.\end{aligned}$$

You can improve on this bound by using the fact that

$$\begin{aligned}\langle \mathbf{1}_S, P(\mathbf{1}_T - \pi(T) \cdot \mathbf{1}) \rangle_\pi &= \langle \mathbf{1}_S - \pi(S) \cdot \mathbf{1}, P(\mathbf{1}_T - \pi(T) \cdot \mathbf{1}) \rangle \\ &\leqslant \|\mathbf{1}_S - \pi(S) \cdot \mathbf{1}\|_\pi \cdot \|P(\mathbf{1}_T - \pi(T) \cdot \mathbf{1})\|_\pi \\ &\leqslant \sqrt{\pi(S) \cdot \pi(T) \cdot (1 - \pi(S)) \cdot (1 - \pi(T))}\,.\end{aligned}$$ ☐

Specialized to d-regular graphs, the expander mixing lemma tells us:

Corollary 2.2. $|E(S,T)| = \frac{d}{n}|S| \cdot |T| \pm \lambda d \sqrt{|S| \cdot |T|}$.

While the proof of the expander mixing lemma is extremely simple, it is quite a powerful statement! Here are a couple corollaries that one may derive.

Corollary 2.3 (Small set edge density)**.** *In a d-regular λ-two-sided spectral expander, for any set S of size at most εn, the average degree of the induced subgraph $G[S]$ is at most $(\lambda + \varepsilon)d$.*

Corollary 2.4 (Vertex expansion)**.** *For any set S of size at most εn in a d-regular λ-two-sided spectral expander, the size of its neighborhood $|N(S)|$ is at least $\frac{(1-\varepsilon d)^2}{\lambda^2}|S|$.*

Proof. Let T be the set of neighbors of S of cardinality $\gamma|S|$. By the expander mixing lemma, we have:

$$d|S| = E(S,T) \leqslant \frac{d}{n}|S| \cdot |T| + \lambda\sqrt{|S| \cdot |T|} \leqslant \frac{d}{n}\gamma|S|^2 + \lambda d\sqrt{\gamma}|S| \leqslant \varepsilon\gamma d|S| + \lambda d\sqrt{\gamma}|S|\,.$$

Using $\lambda < d$, and rearranging, we get $(1 - \varepsilon d)d \leqslant \lambda d\sqrt{\gamma}$, which implies $\gamma \geqslant \left(\frac{1-\varepsilon d}{\lambda}\right)^2$. ☐

2.1 Vignette: Consensus in a spectral expander

Imagine n people who love playing games. Each day, a person plays either ping pong or tennis based on a majority vote of what their friends do. In the long term, what do people play if on day 1, 60% of the people play ping pong? It turns out that if the friendship graph is a λ-two-sided spectral expander, then in only $O(\log n)$ days, everyone switches to playing ping pong.

This is formalized via *majority dynamics*.

Definition 2.5 (Majority dynamics)**.** Suppose G is a graph and $f_0 : V(G) \to \{0,1\}$ is some arbitrary Boolean function. The *majority dynamics* refers to the evolution of the Boolean function as follows. The function f_{i+1} is defined in terms of f_i as follows:

$$f_{i+1}(v) := \mathrm{Maj}\Big((f(u))_{u \in N(v)}\Big)\,.$$

Claim 2.6. Suppose $\mathbf{E}_{u\sim\pi} f_0(u) = p > 1/2 + \lambda$, then after ℓ rounds of majority dynamics, we have:

$$\mathbf{E}_{u\sim\pi} f_\ell(u) \geqslant 1 - (1-p) \cdot \left(\frac{\lambda}{p-\frac{1}{2}}\right)^{\ell}$$

In service of proving this claim, we first show the following fact about spectral expanders.

Lemma 2.7. *Let G be a λ-two-sided spectral expander, and let $T \subseteq V(G)$. Then:*

$$\mathbf{Pr}_{u\sim\pi}[\mathbf{Pr}_{v\sim_G u}[v \in T] < \pi(T) - t\lambda] \leqslant \frac{\pi(T)\cdot(1-\pi(T))}{t^2}.$$

Proof. Let S_t be the set of all vertices u such that $\mathbf{Pr}_{v\sim_G u}[v \in T] < \pi(T) - t\lambda$. The goal is to obtain a bound on $\pi(S_t)$. By the expander mixing lemma (Lemma 2.1) applied to S_t and T,

$$\mathbf{Pr}_{u\sim\pi, v\sim_G u}[u \in S_t, v \in T] \geqslant \pi(S_t)\cdot\pi(T) - \lambda\sqrt{\pi(S_t)\pi(T)(1-\pi(T))},$$

and dividing out by $\pi(S_t)$ gives:

$$\mathbf{Pr}_{u\sim\pi|S_t, v\sim_G u}[v \in T] \geqslant \pi(T) - \lambda\sqrt{\frac{\pi(T)(1-\pi(T))}{\pi(S_t)}}.$$

The LHS can be equivalently written as:

$$\mathbf{E}_{u\sim\pi|S_t}\mathbf{Pr}_{v\sim_G u}[v \in T] < \pi(T) - t\lambda$$

using the definition of S_t. This gives us the inequality:

$$t < \sqrt{\frac{\pi(T)(1-\pi(T))}{\pi(S_t)}},$$

which in particular implies that

$$\pi(S_t) < \frac{\pi(T)\cdot(1-\pi(T))}{t^2}. \qquad \square$$

We are now ready to prove Claim 2.6.

Proof of Claim 2.6. Let $p_s := \mathbf{E}_{u\sim\pi} f_s(u)$, and let $q_s := 1 - p_s$. Then, by Lemma 2.7 with $t = \frac{\pi(T)-\frac{1}{2}}{\lambda}$, we have:

$$q_{s+1} \leqslant \frac{p_s(1-p_s)}{t^2} \leqslant q_s \cdot \frac{\lambda^2}{(p_s - \frac{1}{2})^2} \leqslant q_s \cdot \frac{\lambda^2}{(p_0 - \frac{1}{2})^2},$$

where the third inequality uses monotone-increasing-ness of p_s, which can be derived from the recursion, using $p_0 - 1/2 > \lambda$. By this recursion, we have:

$$q_\ell \leqslant q_0 \cdot \left(\frac{\lambda}{p_0 - \frac{1}{2}}\right)^{2\ell}. \qquad \square$$

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