CS 496 Lecture 4: Spectral Embeddings and Cheeger Rounding

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1 Cheeger's inequalities

Recall the Cheeger inequalities from an earlier lecture, where $\Phi_E(G)$ is the edge expansion or conductance.

Theorem 1.1 (Cheeger inequalities, [AM85]). *The following inequality relating the second eigenvalue and edge expansion is true:*

$$2\Phi_E(G) \geqslant 1 - \lambda_2(P_G) \geqslant \frac{\Phi_E(G)^2}{2}.$$

Recall that we proved the first inequality in an earlier lecture, that a large spectral gap forces good expansion. Today, we will prove the second inequality—i.e. when the spectral gap is small, then the witness eigenvector can be "rounded" to a low conductance set.

Before we get into the proof of the hard direction of Cheeger inequalities, let us discuss an interpretation for what the second eigenvector is doing. Define the *Laplacian* as $L_G = I - P_G$, and the *Dirichlet form* as $\mathcal{E}(f,f) = \langle f, L_G f\rangle_\pi = \mathbf{E}_{uv\sim G}(f(u) - f(v))^2$.

The eigenvector v_2 corresponding to the eigenvalue λ_2 can then be interpreted as:

$$v_2 = \underset{f:V(G)\to\mathbb{R},\, f \text{ nonconstant}}{\arg\min} \frac{\mathcal{E}(f,f)}{\mathbf{Var}_\pi f}.$$

Rounding via threshold cuts. Let f be a (real) eigenvector for λ_2 with $\mathbb{E}_\pi f = 0$ and $f \not\equiv 0$. We may scale f so that it lives in an interval of width 1.

To round f to a partition of the vertices of the graph with a low conductance cut in between, we find a threshold τ and partition the graph based on whether a vertex lies to the left or right of τ.

For the sequel, for $\tau \in [0,1]$, define $S_\tau \coloneqq \{u : f(u) \geqslant \tau\}$.

We will show the existence of a threshold that finds a low conductance cut via a probabilistic argument. More concretely,

$$\begin{aligned}\min_\tau \Phi_E(S_\tau) &= \min_\tau \frac{\mathbf{Pr}_{uv\sim E(G)}[u \in S_\tau, v \notin S_\tau]}{\pi(S_\tau)} \\ &\leqslant \min_\tau \frac{\mathbf{Pr}_{uv\sim E(G)}[u \in S_\tau, v \notin S_\tau]}{\pi(S_\tau)(1-\pi(S_\tau))}\end{aligned}$$

$$\leqslant \frac{\mathbf{E}_{\tau\sim\mathcal{D}}\mathbf{Pr}_{uv\sim E(G)}[u \in S_\tau, v \notin S_\tau]}{\mathbf{E}_{\tau\sim\mathcal{D}}\pi(S_\tau)(1-\pi(S_\tau)}$$

where the last inequality uses the fact that $\frac{\sum_i a_i}{\sum_i b_i} \geqslant \min_i \frac{a_i}{b_i}$ as long as (a_i) and (b_i) are nonnegative sequences.

Thus, we need to:

- Design a distribution $\mathcal{D}$.
- Analyze the numerator and denominator of the above expression.

It turns out that a good distribution for us to use is the distribution with density $p_{\mathcal{D}}(t) = 2|t|$.

We now analyze the cut size, and the set size.

Expected numerator. We can analyze the numerator as follows. First observe:

$$\mathbf{E}_{\tau\sim\mathcal{D}}\mathbf{Pr}_{uv\sim E(G)}[u \in S_\tau, v \notin S_\tau] = \mathbf{E}_{uv\sim E(G)}\mathbf{Pr}_{\tau\sim\mathcal{D}}[u \in S_\tau, v \notin S_\tau]$$

For a fixed choice of u and v, observe that

$$\mathbf{Pr}_{\tau\sim\mathcal{D}}[u \in S_\tau, v \notin S_\tau] = \left|\operatorname{sign}(f(v)) \cdot f(v)^2 - \operatorname{sign}(f(u)) \cdot f(u)^2\right|.$$

Thus,

$$\begin{aligned}
\mathbf{E}_{uv\sim E(G)}\mathbf{Pr}_{\tau\sim\mathcal{D}}[u \in S_\tau, v \notin S_\tau] &= \mathbf{E}_{uv\sim E(G)}\left|\operatorname{sign}(f(v)) \cdot f(v)^2 - \operatorname{sign}(f(u)) \cdot f(u)^2\right| \\
&\leqslant \mathbf{E}_{uv\sim E(G)}|f(u)-f(v)| \cdot (|f(u)|+|f(v)|) \\
&\leqslant \sqrt{\mathbf{E}_{uv\sim E(G)}(f(u)-f(v))^2 \cdot \mathbf{E}_{uv\sim E(G)}(|f(u)|+|f(v)|)^2} \\
&\leqslant 2\sqrt{\mathcal{E}(f,f)\cdot \mathbf{Var}_\pi[f]}.
\end{aligned}$$

Expected denominator. Note that

$$\pi(S_\tau)\big(1-\pi(S_\tau)\big) = \mathbf{Pr}_{u,v\sim\pi}[u \in S_\tau, v \notin S_\tau].$$

Taking expectation over τ gives:

$$\mathbf{E}_{\tau\sim\mathcal{D}}\pi(S_\tau)\big(1-\pi(S_\tau)\big) = \frac{1}{2}\mathbb{E}_{u,v\sim\pi}\left|\operatorname{sign}(f(u)) \cdot f(u)^2 - \operatorname{sign}(f(v)) \cdot f(v)^2\right| \geqslant \frac{1}{2}\mathbf{E}_{u,v\sim\pi}(f(u)-f(v))^2,$$

which is equal to $\mathbf{Var}_\pi f$.

Thus, we get that:

$$\min_\tau \Phi(S_\tau) \leqslant \frac{2\sqrt{\mathcal{E}(f,f)\cdot\mathbf{Var}_\pi f}}{\mathbf{Var}_\pi f} = 2\sqrt{1-\lambda_2}$$

In particular, we have proved $\frac{\Phi_E(G)^2}{4} \leqslant 1-\lambda_2$.

Remark 1.2. The tight examples for Cheeger are described by the hypercube and the sphere. The second eigenvector space of the d-dimensional Boolean hypercube is the set of all linear functions, and the spectral gap can be calculated to be $1/d$. The sparsest cut can also be showed to be a "coordinate cut", i.e., choose S as the set of all $x \in \{0,1\}^d$ such that $x(1) = 0$. The conductance of the coordinate cut can also be seen to equal $1/d$, which essentially shows that the "easy direction" of Cheeger can essentially be made tight.

The sphere witnesses the tightness for the other direction. The eigenvectors are yet again linear functions, and the sparsest cut is a "halfspace cut", i.e., the set $S \subseteq \mathbb{S}^{d-1}$ such that $x(1) \geqslant 0$.

References

[AM85] Noga Alon and Vitali D Milman. $\lambda 1$, isoperimetric inequalities for graphs, and superconcentrators. *Journal of Combinatorial Theory, Series B*, 38(1):73–88, 1985. 1