CS 496 Lecture 6: Alon–Boppana Bound and Ramanujan Graphs

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1 Limits on expansion

Let G be a d-regular graph with adjacency matrix A_G. Denote the eigenvalues by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$, and in particular $\lambda_1(G) = d$. The main motivating question in today's lecture is: how small can $\lambda_2(G)$ be?

Theorem 1.1 (Alon–Boppana bound [Nil91])**.** *For every fixed $d \geqslant 3$ and any n-vertex d-regular graph G, we have:*

$$\lambda_2(G) \geqslant 2\sqrt{d-1} - o_n(1)\,.$$

Today, we will prove Theorem 1.1, but in a slightly circuitous way that hopefully gives a sense of the lay of the land.

So, where does this number $2\sqrt{d-1}$ come from? It turns out that this is exactly equal to the spectral radius of the *d-regular infinite tree T_d*. This is no coincidence. Indeed, the proof that we will see today will directly connect the spectral radius of the infinite tree to the expansion of a finite d-regular graph.

A notion that shows up time and again in the study of expansion of finite graphs is that of a *cover* or a *lift*.

Definition 1.2 (Cover)**.** A graph H *covers* a graph G if there is a map $\phi : V(H) \to V(G)$ such that

(i) for every edge $uv \in E(H)$, $\phi(u)\phi(v) \in E(G)$; and

(ii) for every $u \in V(H)$, the restriction $\phi : N_H(u) \to N_G(\phi(u))$ is a bijection.

Remark 1.3. A cover is sometimes called a *lift*. One may also think of forming H from G by replacing each vertex of G with a "cloud" of r vertices (where $r = |V(H)|/|V(G)|$) and replacing each edge uv with a perfect matching between the two corresponding clouds.

Observe that T_d is a cover of *every* d-regular graph, which earns it its title as the *universal cover*.

2 Spectral radius of T_d

Let A_T denote the adjacency operator on $\ell^2(T_d)$: the set of all functions $f : V(T_d) \to \mathbb{R}$ such that $\sum_{v \in V(T_d)} |f(v)|^2 < \infty$; it can be verified that this is a well-defined inner product space.

Its operator norm is

$$\|A\| = \sup_{f \in \ell^2(T_d) \setminus 0} \frac{\|Af\|_2}{\|f\|_2}.$$

We will show $\|A\| = 2\sqrt{d-1}$ by counting closed walks (the *trace method*). Connecting the closed walk counts to the operator norm of the infinite tree takes a little bit of functional analysis to formalize, but we will skip that since that isn't the point of the class. Instead, let's see how trace powers relate to the spectral radius of a finite matrix. In the sequel we will use $\rho(M)$ to denote the spectral radius of a matrix M.

Observation 2.1. For any symmetric matrix M and integer ℓ, we have: $\rho(M) = \lim_{\ell \to \infty} \left(\operatorname{tr} M^{2\ell}\right)^{1/(2\ell)}$.

Proof. Observe that $\operatorname{tr} M^{2\ell} = \sum_{i=1}^n \lambda_i^{2\ell} \in [\rho(M)^{2\ell}, n \cdot \rho(M)^{2\ell}]$, and in particular $\left(\operatorname{tr} M^{2\ell}\right)^{1/(2\ell)} \in \rho(M) \cdot [1, n^{1/(2\ell)}] \to \rho(M)$ as $\ell \to \infty$. $\square$

It turns out that for the d-regular infinite tree, one can get a handle on its spectral radius by looking at a single diagonal entry of $A_T^{2\ell}$, which precisely counts the number of closed walks of length-2ℓ starting and ending at a fixed vertex v in T_d.

Counting closed walks on T_d. One may draw the infinite tree by first setting some vertex v as the root, and then draw 1 neighbor "up" and $d-1$ neighbors "down", and extend this picture indefinitely.

A closed walk from the root consists of steps that either move *up* or *down*. There are at most $2^{2\ell}$ choices for which steps are up and which steps are down. If a step is taking you away from the root, then there are $(d-1)$ choices for where to walk, while a step towards the root is unique. Since the number of steps away from the root and towards the root must be equal, exactly half the steps are away from the root. Thus, the number of closed walks of length 2ℓ is at most:

$$2^{2\ell}(d-1)^{\ell}.$$

and in particular we obtain

$$\rho(A_T) = 2\sqrt{d-1}.$$

Thus $\|A_T\| = 2\sqrt{d-1}$.

Proving the Alon–Boppana bound. We are now ready to prove the Alon–Boppana bound. Let $\gamma := \|A_T\|$ (so $\gamma = 2\sqrt{d-1}$). We show that for any d-regular G on n vertices,

$$\lambda_2(G) \geqslant \gamma - o_n(1).$$

The cool aspect of the proof is that it is agnostic to the exact value of γ—it simply translates the infinite tree's spectral radius to the second eigenvalue of a finite matrix.

Setup. Pick $f \in \ell^2(T_d)$ with $\|f\|_2 = 1$ and $\|Af\|_2 \geqslant \gamma - \varepsilon$. Truncate f to a finitely supported function $\widetilde{f}$ so that $\|\widetilde{f}\|_2 \geqslant 1 - \varepsilon$ and $\|A\widetilde{f}\|_2 \geqslant \gamma - 2\varepsilon$; the support of $\widetilde{f}$ lies in a finite ball of the tree.

Paste into G. Choose two vertices $u, v \in V(G)$ whose radius-R neighborhoods are disjoint, with $R \to \infty$ as $n \to \infty$ (e.g., $R = \frac{1}{2}\log_{d-1} n$ works for almost all d-regular graphs and for many explicit families). Use the covering map from those balls to paste copies of $\widetilde{f}$ into G, supported on the two disjoint balls around u and v (if multiple vertices from $\mathrm{supp}(\widetilde{f})$ map to the same vertex x, we place the ℓ_2 norm of the vector of values assigned to the preimage of x; but for now, let's ignore this consideration and pretend that the neighborhoods look tree-like). Let these be f_u and f_v.

Because supports are far apart and A is local, Af_u and Af_v are orthogonal; moreover

$$\frac{\langle f_u, Af_u\rangle}{\|f_u\|_2^2}, \frac{\langle f_v, Af_v\rangle}{\|f_v\|_2^2} \geqslant \gamma - 2\varepsilon - o_R(1).$$

An upshot of the above is that for every function g in the 2D subspace $\mathcal{U} = \mathrm{span}\{f_u, f_v\}$, the quadratic form is at least $\gamma - O(\varepsilon)$.

3 Ramanujan graphs

Definition 3.1. A finite d-regular graph G is *Ramanujan* if every nontrivial eigenvalue $\lambda \in \mathrm{Spec}(A) \setminus \{d\}$ satisfies $|\lambda| \leqslant 2\sqrt{d-1}$. Equivalently, the new eigenvalues of any nontrivial lift lie within the tree spectrum.

There are 3 widely different techniques that let us get a handle on Ramanujan graphs, and we will see some of the ideas from each one in the coming few lectures.

- **Explicit Ramanujan graphs for $d = p + 1$** (prime p): Lubotzky–Phillips–Sarnak [LPS88] and Margulis [Mar88] constructed Ramanujan graphs using Cayley graphs of arithmetic groups.
- **Explicit Ramanujan graphs for $d = q + 1$** (prime powers q): Morgenstern [Mor94] extended to all prime powers.
- **Existence of bipartite Ramanujan graphs for all d:** Marcus–Spielman–Srivastava [MSS15] proved the existence of infinite families where all nontrivial eigenvalues are less than $2\sqrt{d-1}$ using a new technique they pioneered called *interlacing polynomials*, which also lead to a major breakthrough in the resolution of the Kadison-Singer conjecture.
- **Random d-regular graphs:** Alon conjectured that a random d-regular is asymptotically Ramanujan; Friedman proved that $\lambda_2 \leqslant 2\sqrt{d-1} + o(1)$ with high probability [Fri08]. Later, Bordenave [Bor19] gave a simpler proof. Even more recently, due to a striking work of Huang–McKenzie–Yau [HMY24], we know that a random d-regular graph is truly Ramanujan graph occur with probability ≈ 0.69.

References

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