

CS 496 Lecture 7: Near-Ramanujan graphs via random graphs and lifts

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1 Models of random regular graphs

Let G be an n -vertex d -regular graph for $d \geq 3$, and let A_G be the unnormalized adjacency matrix of G .

Theorem 1.1 ([Fri08]). *Let G be a random d -regular graph on n vertices. Then, with probability $1 - o_n(1)$,*

$$\lambda_2(A_G) \leq 2\sqrt{d-1} \cdot (1 + o_n(1)).$$

Equivalently, G is near-Ramanujan with high probability.

A concrete model for sampling a random regular graph is the *configuration model*.

Definition 1.2 (Configuration model). Fix n and d with nd even. Form a multigraph on vertex set $[n]$ as follows: attach d labeled half-edges to each vertex $v \in [n]$, and choose a uniformly random perfect matching on the nd half-edges. For each matched pair, identify the endpoints to form an edge.

While the configuration model may produce self-loops or multi-edges, conditioning on simplicity yields the uniform distribution over simple d -regular graphs.

In today's lecture, we will see how to *derandomize* an appropriate version of Friedman's theorem and obtain a deterministic $\text{poly}(n)$ -time algorithm for constructing near-Ramanujan graphs.

Derandomization at a high level. A proof of [Theorem 1.1](#) given by [Theorem 2.3](#) can be verified to succeed if the graph G is drawn from any distribution $\mathcal{D}$ that matches the configuration model on $\Theta(\log n)$ moments. Very roughly, this is because the proof proceeds by the *trace method*, i.e. controlling variants of $\mathbb{E} \text{tr } A_G^\ell$.¹

This suggests the following route to an explicit construction:

If a pseudorandom generator can produce a distribution over matchings of the nd half-edges that *matches* the relevant moments (up to $k = \log n$), then the near-Ramanujan conclusion still goes through. Consequently, one may brute-force over all seeds of such a PRG in $\text{poly}(n)$ time and output a near-Ramanujan graph.

¹ The literal trace moments are not bounded due to some ill-behaved tail events, i.e., when the random graph has large *bicycles*, so appropriate care needs to be taken. We ignore the associated subtleties in this discussion.

Matching k moments typically requires k -wise independence, or a close approximation to it. If we take $k = \Theta(\log n)$, then a k -wise uniform construction of permutation costs $\Theta(k \log n) = \Theta((\log n)^2)$ random bits, which is too expensive for our endgame if we hope to search over all seeds in polynomial time.

Remark 1.3. Constructions of k -wise uniform families of permutations on $[n]$ using $\Theta(k \log n)$ random bits were given by Kaplan, Naor, and Reingold [KNR09], and also by Kassabov [Kas07]. On the other hand, a simple entropy argument proves that $k \log n$ bits are actually necessary. It turns out that even if one is willing to tolerate some tiny error in k -wise independent (i.e. require that degree- k moments only *approximately* match the true uniform distribution), one needs $k \log n$ bits. This is because a result of Alon and Lovett [AL13] proves that an “approximately” k -wise uniform distribution over permutations is close in total variation distance to a truly k -wise uniform distribution over permutations.

Starting point: Generate a moderately sized graph with few random bits. Consider the target size $n_0 := \exp(\sqrt{\log n})$. There is a $\log n_0$ -wise uniform distribution over permutations on dn_0 , that uses $(\log n_0)^2 = \log n$ random bits, so one may brute force a good seed in $\text{poly}(n)$ time to explicitly find an n_0 -vertex d -regular graph that is near-Ramanujan. In fact, one may also guarantee that the constructed graph has no big *bicycles*, and is simple.

The remaining task is to amplify the graph from size n_0 up to size n while preserving near-Ramanujan property. This is where *graph lifts* enter the picture.

2 Graph lifts

Definition 2.1 (Graph lift). Let $G = (V, E)$ be a graph and let $r \in \mathbb{N}$. An r -lift of G is a graph $\tilde{G}$ obtained by replacing each vertex $v \in V$ by a *cloud* $\{(v, i) : i \in [r]\}$ and, for every edge $uv \in E$, placing a perfect matching between the clouds $\{(u, i)\}_{i \in [r]}$ and $\{(v, i)\}_{i \in [r]}$. If G is d -regular, then every r -lift $\tilde{G}$ is also d -regular and has $|V(\tilde{G})| = r |V(G)|$.

Definition 2.2 (Random lift). Fix G and r . A *random r -lift* is obtained by choosing, independently for each $uv \in E(G)$, a uniformly random perfect matching between the two clouds attached to u and v .

Random lifts scale up the graph while preserving expansion and spectral properties, as articulated by the following result of Bordenave [Bor19].

Theorem 2.3. Let G be a fixed d -regular base graph and let $\tilde{G}$ be a random r -lift. As $r \rightarrow \infty$,

$$\lambda_2(A_{\tilde{G}}) \leq 2\sqrt{d-1} + o_r(1) \quad \text{with probability } 1 - o_r(1).$$

While Theorem 2.3 is excellent news qualitatively, drawing a truly random r -lift does not, by itself, significantly reduce the number of random bits: the independent perfect matchings along each edge carry substantial entropy.

Random 2-lifts on bicycle-free graphs. Following Mohanty, O’Donnell, Paredes [MOP20], we focus on 2-lifts (the case $r = 2$), which can be encoded by assigning a signing $\sigma : E(G) \rightarrow \{\pm 1\}$

to the edges of G . The matching between the two clouds along uv is determined by $\sigma(uv)$ —it is $\{((u, +), (v, +)), ((u, -), (v, -))\}$ if $\sigma(uv) = +1$, and is $\{((u, +), (v, -)), ((u, -), (v, +))\}$ if $\sigma(uv) = -1$. The eigenvalues of a 2-lift of G are characterized by the eigenvalues of G itself along with new eigenvalues that are precisely given by the eigenvalues of a *signed* adjacency matrix associated to σ .

We now describe the notion of a *bicycle*, necessary to describe the technical condition on graphs whose random 2-lifts [MOP20] can analyze.

Definition 2.4 (Bicycles). An ℓ -bicycle is a connected subgraph on at most ℓ -edges that contains at least two cycles. We say a graph is ℓ -bicycle-free if it has no bicycles of length $\leq \ell$.

Remark 2.5. The terminology *bicycle* is slightly non-standard, and is only used in a handful of papers. The more common term in the literature is *tangle*.

Theorem 2.6 ([MOP20]). Let G be a d -regular base graph with no ℓ -bicycles for $\ell = \Theta((\log \log |V(G)|)^2)$. Let $\tilde{G}$ be a uniformly random 2-lift of G . Then $\tilde{G}$ is near-Ramanujan with high probability.

This is particularly useful in our pipeline because when we derandomize the configuration model at the moderate size $n_0 = \exp(\sqrt{\log n})$, we can, by moment-y arguments, also ensure the produced base graph has no ℓ -bicycles for $\ell = O(\sqrt{\log n})$.

Seed-length and exhaustive search for lifts. The analysis in [MOP20] of random 2-lifts proceeds via control of low moments of the signed non-backtracking operator; consequently it only requires *approximately* $\log n$ -wise independence among the edge signs $\sigma(uv)$. There are PRGs with seed length $O(\log n)$ that achieve this level of independence to the accuracy required by the moment bounds (see, e.g., [NN93]). Therefore:

Algorithm 2.7 (Brute-force over short seeds). For every n and d , there exists a PRG $G: \{0, 1\}^{O(\log n)} \rightarrow \{\pm 1\}^{E(G)}$ such that, for every d -regular G with no large bicycles, the 2-lift determined by the edge-signs $\sigma = G(s)$ is near-Ramanujan for *some* seed s .

Since the seed space has size $n^{O(1)}$, we can deterministically find such an s in polynomial time by exhaustive search.

Scaling to n via iterated lifts. Starting from the moderate-size near-Ramanujan base G_{n_0} (with no small bicycles), we iteratively apply Algorithm 2.7:

$$G_{n_0} \xrightarrow{\text{2-lift}} G_{2n_0} \xrightarrow{\text{2-lift}} G_{4n_0} \xrightarrow{\cdots} G_n,$$

each time searching over $n^{O(1)}$ seeds to locate a near-Ramanujan 2-lift. After $O(\log(n/n_0)) = O(\log n)$ stages, we obtain a d -regular n -vertex graph G_n with

$$\lambda_2(A_{G_n}) \leq 2\sqrt{d-1}(1+o(1)),$$

constructed deterministically in $\text{poly}(n)$ time.

Remark 2.8 (The need for a certificate). The search over seeds must be coupled with a fast procedure to check that the seed is actually good. The fact that eigenvalues of a matrix can be computed in polynomial time is a crucial ingredient to get this procedure to succeed. The lack of a certificate is exactly what precludes using this strategy to construct other kinds of expanders, such as *vertex expanders*.

Analyzing the spectrum of a random 2-lift. In the next piece, we dive into the [MOP20] argument.

Given a 2-lift of G encoded by the signing σ , let A_σ be the *signed adjacency matrix* defined by

$$(A_\sigma)_{uv} = \begin{cases} \sigma(uv) & \text{if } uv \in E, \\ 0 & \text{otherwise.} \end{cases}$$

It is standard that the spectrum of the 2-lift $\tilde{G}$ is the multiset union of the spectrum of A_G (the “old” eigenvalues) and the spectrum of A_σ (the “new” eigenvalues). Consequently, to show that a *random* 2-lift is near-Ramanujan, it suffices to control the operator norm of A_σ .

Lemma 2.9 (Reduction to the signed matrix). *Let $\tilde{G}$ be the 2-lift of G determined by σ . Then*

$$\text{Spec}(A_{\tilde{G}}) = \text{Spec}(A) \uplus \text{Spec}(A_\sigma).$$

In particular, if $\|A_\sigma\| \leq 2\sqrt{d-1} + o(1)$, then all new eigenvalues lie in the Ramanujan interval $[-2\sqrt{d-1} - o(1), 2\sqrt{d-1} + o(1)]$, and the lift is near-Ramanujan.

To prove a bound on $\|A_\sigma\|$ and hence [Theorem 2.6](#), we will use the *trace method*.

Trace-method setup. Write $\lambda_{\max}(M)$ for the spectral radius of a symmetric matrix M . For any even ℓ ,

$$\lambda_{\max}(A_\sigma)^\ell \leq \text{tr } A_\sigma^\ell.$$

An upper bound on $\mathbb{E}_\sigma \text{tr } A_\sigma^\ell$ of the form $(2\sqrt{d-1} + o(1))^\ell$ for a suitably growing even ℓ already implies by Markov that $\|A_\sigma\| \leq 2\sqrt{d-1} + o(1)$ with high probability over the random signs.

To understand this quantity, we expand the trace in the standard way:

$$\text{tr } A_\sigma^\ell = \sum_{v_0 \in V} \sum_{v_1, \dots, v_{\ell-1}} (A_\sigma)_{v_0 v_1} (A_\sigma)_{v_1 v_2} \cdots (A_\sigma)_{v_{\ell-1} v_0}.$$

Each length- ℓ term corresponds to a closed walk $w = (v_0, v_1, \dots, v_{\ell-1}, v_0)$ in G , contributing the product $\prod_{e \in E} \sigma(e)^{m_e(w)}$, where $m_e(w)$ is the number of times w traverses edge e (in either direction). Taking expectation over the *independent, uniform* signs $\sigma(e) \in \{\pm 1\}$ gives:

$$\mathbb{E}_\sigma \left[\prod_e \sigma(e)^{m_e(w)} \right] = \begin{cases} 1 & \text{if } m_e(w) \text{ is even for every } e, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathbb{E}_\sigma \text{tr } A_\sigma^\ell = \#\{\text{closed length-}\ell \text{ walks in } G \text{ that use every edge an even number of times}\}. \quad (1)$$

Heuristic: tree-like walks dominate. The right-hand side of (1) is a purely *combinatorial* quantity. The guiding intuition is that the dominant contribution comes from walks that are *tree-like*: they live on portions of G that look like the d -regular infinite tree T_d . On an actual tree, the constraint “every edge is used an even number of times” is automatically enforced by the walk being closed.

Last lecture, we saw how to count closed walks in the d -regular infinite tree. Given our guiding intuition, one expects

$$\mathbb{E}_\sigma \text{tr } A_\sigma^\ell \lesssim n \cdot (2\sqrt{d-1})^\ell$$

which in turn suggests

$$\left(\mathbf{E}_\sigma \operatorname{tr} A_\sigma^\ell\right)^{1/\ell} \lesssim 2\sqrt{d-1} + o(1).$$

This then yields

$$\|A_\sigma\| \leq 2\sqrt{d-1} + o(1) \quad \text{with high probability over the random signs.}$$

References

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