CS 496 Lecture 8: Zig-Zag Product

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Last time we saw Friedman's theorem and how to derandomize it to get an explicit construction of expanders. Today we'll see a much simpler, fully elementary construction—the *zig–zag product* of Reingold, Vadhan, and Wigderson [RVW02]. The payoff is that with a simple gadget we can *grow* a small expander into a large one while maintaining a bound on the degree.

For a graph X, let P_X denote its random-walk matrix, and let $\lambda_X := \max\{\lambda_2(P_X), -\lambda_n(X)\}$.

1 The Zig–Zag Product

Let G be a D-regular graph on vertex set $V(G)$ with $|V(G)| = n$. Let H be a d-regular graph on vertex set $[D] = \{1, \ldots, D\}$. Further, assume that for every vertex v in $V(G)$, there is some ordering on its D incident edges and neighbors.

Vertex set and degree. The zig-zag product $Z := G \bowtie H$ has vertex set

$$V(Z) = \{(v,i) : v \in V(G),\ i \in [D]\},$$

i.e., each $v \in V(G)$ is replaced by a *cloud* of D vertices $(v,1), \ldots, (v,D)$. Every vertex in Z has degree d^2: a step in Z is the composition

$$(\textit{zig} \text{ in the cloud via } H) \;\to\; (\text{swap along } G) \;\to\; (\textit{zag} \text{ in the cloud via } H).$$

Edge rule (length-3 walk). Formally, for each (v,i):

1. (**zig**) Walk to any (v,j) such that ij is an edge in H.
2. (**swap**) Walk from (v,j) to (u,j') where u is the j-th neighbor of v, and v is the j'-th neighbor of u.
3. (**zag**) Walk from (u,j') to (u,k) for any neighbor k of j' in H.

Thus neighbors in Z correspond exactly to length-3 walks of the form

$$(v,i) \xrightarrow{H} (v,j) \xrightarrow{G} (u,j') \xrightarrow{H} (u,k).$$

Hence, Z is d^2-regular, and $|V(Z)| = nD$.

1.1 Expansion of Z

It remains to analyze the spectral expansion of Z. Index $V(Z)$ as pairs (v, i) with $v \in V(G)$ and $i \in [D]$. Let Id_n denote the identity on functions on $V(G)$. Define:

- P_H as the random walk matrix of H.
- The matching S_G on pairs: $S_G(v, i) = (u, j')$ if u is the i-th neighbor of v in G and v is the j'-th neighbor of u. Then $S_G = S_G^{-1} = S_G^{\top}$.

Then the normalized adjacency of the zig-zag product is

$$P_Z = (P_H \otimes \mathrm{Id}_n)\, S_G\, (P_H \otimes \mathrm{Id}_n).$$

We can write P_H as $\frac{1}{D}11^{\top} + \underline{P}_H$ where $\|\underline{P}_H\| \leqslant \lambda_H$. Essentially, the point of this is that we can write P_Z, up to very small spectral error, as $(J \otimes \mathrm{Id}_n)\, S_G\, (J \otimes \mathrm{Id}_n)$, where we are abbreviating $\frac{1}{D}11^{\top}$ as J. Concretely, we can write P_Z as:

$$(J \otimes \mathrm{Id}_n)\, S_G\, (J \otimes \mathrm{Id}_n) + (J \otimes \mathrm{Id}_n)\, S_G\, (\underline{P}_H \otimes \mathrm{Id}_n) + (\underline{P}_H \otimes \mathrm{Id}_n)\, S_G\, (P_H \otimes \mathrm{Id}_n)$$

The second and third terms are at most λ_H in spectral norm, and so we have:

$$\|P_Z - (J \otimes \mathrm{Id}_n)\, S_G\, (J \otimes \mathrm{Id}_n)\| \leqslant 2\lambda_H\,.$$

Thus, to understand the expansion of P_Z, it suffices to understand the behavior of

$$(J \otimes \mathrm{Id}_n)\, S_G\, (J \otimes \mathrm{Id}_n)\, f$$

for f orthogonal to 1.

It is easy to verify that the above is equal to:

$$1_D \otimes \left(P_G \widetilde{f}\right),$$

where $\widetilde{f}$ is the n-dimensional vector where the u-th entry contains the mean of f on cloud u. The norm of this vector is $\sqrt{D}\left\|\widetilde{f}\right\| \cdot \lambda_G \leqslant \lambda_G \cdot \|f\|$.

Putting the above together tells us:

$$\lambda_2(P_Z) \leqslant \lambda_G + 2\lambda_H\,.$$

Keeping Degree Fixed and Growing Size

Notice that Z is d^2-regular regardless of D. To get an n-sized construction starting from smaller graphs, we need to repeatedly apply the zig-zag product, and ensure that the expansion does not deteriorate.

The concrete way we achieve this given a gadget H is via the following routine:

$$G \quad \longmapsto \quad G^2 \text{ (square)} \quad \longmapsto \quad Z \;=\; G^2 \bowtie H.$$

Since $\lambda(G^2) = \lambda(G)^2$, applying the zig-zag bound gives the recurrence

$$\lambda_{\text{next}} \leqslant \lambda_{\text{current}}^2 + 2\,\lambda(H).$$

With $\lambda(H) \leqslant 2/\sqrt{d}$ (if H is Ramanujan), this becomes

$$\boxed{\lambda_{i+1} \leqslant \lambda_i^2 + \frac{4}{\sqrt{d}}}.$$

Suppose d is a sufficiently large constant, and $\lambda_i \leqslant \frac{8}{\sqrt{d}}$, then the above recurrence guarantees $\lambda_{i+1} \leqslant \frac{8}{\sqrt{d}}$ too. While the graph is quite far from being Ramanujan, since it is d^2-regular but has second eigenvalue $O\left(\frac{1}{\sqrt{d}}\right)$, it is still a pretty good expander!

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002. 1