CS 496 Homework

Due Date: June 10

Problemset

Problem 1. Let $\mathcal{D}$ be an n-dimensional isotropic distribution, i.e., $\mathbf{E}_{x\sim\mathcal{D}}\boldsymbol{x}\boldsymbol{x}^\top = \mathrm{Id}$. Prove that there exists a set $S \subseteq \mathrm{supp}(\mathcal{D})$ such that:

1. For any distinct $x, y \in S$, $\|x - y\|_2 \geqslant \Omega(\sqrt{n})$.
2. $|S| \geqslant \Omega(n)$.

Problem 2. Let $\boldsymbol{A} \sim \mathcal{N}(0,1)^{n\times m}$ and suppose that $m \leqslant \alpha n$ for a sufficiently small absolute constant $\alpha > 0$. The goal of this problem is to prove that, with high probability,

$$\sigma_{\min}(A) \geqslant \Omega(\sqrt{n}) \qquad \text{and} \qquad \sigma_{\max}(A) \leqslant O(\sqrt{n}).$$

Recall that

$$\sigma_{\max}(A) = \sup_{x\in\mathbb{S}^{m-1}} \|Ax\|_2, \qquad \sigma_{\min}(A) = \inf_{x\in\mathbb{S}^{m-1}} \|Ax\|_2,$$

where $\mathbb{S}^{m-1} = \{x \in \mathbb{R}^m : \|x\|_2 = 1\}$.

1. **ε-nets.** Let $T \subseteq \mathbb{R}^d$. An ε-net for T is a set $\mathcal{N} \subseteq T$ such that for every $x \in T$, there exists $y \in \mathcal{N}$ satisfying
$$\|x - y\|_2 \leqslant \varepsilon.$$
Prove that for every $0 < \varepsilon < 1$, the unit sphere $\mathbb{S}^{d-1}$ has an ε-net $\mathcal{N}$ with
$$|\mathcal{N}| \leqslant \left(\frac{C}{\varepsilon}\right)^d$$
for some absolute constant $C > 0$.

2. **Upper bound from a net.** Let $\mathcal{N}$ be an ε-net of $\mathbb{S}^{m-1}$. Prove that if
$$\|Ax\|_2 \leqslant B\sqrt{n} \qquad \text{for every } x \in \mathcal{N},$$
then
$$\sigma_{\max}(A) \leqslant \frac{B}{1-\varepsilon}\sqrt{n}.$$
In particular, for any fixed $\varepsilon < 1$, this gives
$$\sigma_{\max}(A) \leqslant O(\sqrt{n}).$$

3. **Lower bound from a net.** Let $\mathcal{N}$ be an ε-net of $\mathbb{S}^{m-1}$. Prove that if

$$\|Ax\|_2 \geqslant b\sqrt{n} \qquad \text{for every } x \in \mathcal{N},$$

and additionally

$$\sigma_{\max}(A) \leqslant B\sqrt{n},$$

then

$$\sigma_{\min}(A) \geqslant (b - \varepsilon B)\sqrt{n}.$$

Conclude that if $\varepsilon > 0$ is chosen sufficiently small as a function of b and B, then

$$\sigma_{\min}(A) \geqslant \Omega(\sqrt{n}).$$

4. **Union bound over the net.** Let $\mathcal{N}$ be an ε-net of $\mathbb{S}^{m-1}$ with

$$|\mathcal{N}| \leqslant \left(\frac{C}{\varepsilon}\right)^m.$$

Use concentration of the norm of a gaussian vector and a union bound over $\mathcal{N}$ to prove that, with probability at least $1 - e^{-\Omega(n)}$,

$$b\sqrt{n} \leqslant \|Ax\|_2 \leqslant B\sqrt{n} \qquad \text{for every } x \in \mathcal{N},$$

provided that $\alpha > 0$ is chosen sufficiently small.

5. **Conclusion.** Combine the previous parts to prove that for some sufficiently small absolute constant $\alpha > 0$, if $m \leqslant \alpha n$, then with probability at least $1 - e^{-\Omega(n)}$,

$$\sigma_{\min}(A) \geqslant \Omega(\sqrt{n}) \qquad \text{and} \qquad \sigma_{\max}(A) \leqslant O(\sqrt{n}).$$

Final Project

For the final project for this class, read any paper(s), parts of a textbook, or survey that is relevant to any of the topics we covered in class, and write a report about the paper.

The report should:

- Describe the landmark results in the area that your reading is related to.
- Describe the contribution of the paper, and a high-level description of the proof.
- Identify one technical idea in the paper you find cool, and articulate it by writing a lemma/theorem statement along with a proof. It could be one ingredient in the proof you find cool, or the core technical idea simplified to communicate the essence without technical details.
- Identify an open problem, and write about why you find it interesting.

As long as you can justify the connection between the paper and the theme of the class, any paper that was not already covered in class should be fine!

You may also attempt a research problem. If you solve it you can just write the problem statement and solution. If you don't (which is totally okay!), your report should:

- Describe a concrete problem statement that you are attempting to solve.
- Talk about the significance of the problem, and explain what makes it interesting.
- Identify key prior work and describe the ideas and contributions of those.
- Describe the methods you attempted to solve the problem, and bottlenecks you encountered. The more concrete the better! For example: a special case you could not resolve; a lemma you tried to prove but failed at; a counterexample to your initial approach; a computer simulation showing why a method may or may not work.

Please turn in your report via email by **June 10**.

Candidate papers

Here is a (very non-exhaustive) list of candidate papers and textbooks:

- A recent breakthrough resolution of Talagrand's convexity conjecture: [HST26].
- Sinho Chewi's textbook on log-concave sampling: `https://chewisinho.github.io/main.pdf`.
- A unified framework for proving mixing times of Markov chains and many applications: [CE22].
- See [BJ25] and some of the papers it cites for some recent great papers in discrepancy theory.
- A very cool approximate John's theorem: [BK25].
- Improved sphere packing with Brownian motion: [Kla26].
- A preceding improvement using probabilistic method techniques: [CJMS23].

References

[BJ25] Nikhil Bansal and Haotian Jiang. Decoupling via Affine Spectral-Independence: Beck-Fiala and Komlós Bounds Beyond Banaszczyk. *arXiv preprint arXiv:2508.03961*, 2025. 3

[BK25] Pierre Bizeul and Boaz Klartag. Distances between non-symmetric convex bodies: optimal bounds up to polylog. *arXiv preprint arXiv:2510.20511*, 2025. 3

[CE22] Yuansi Chen and Ronen Eldan. Localization schemes: A framework for proving mixing bounds for markov chains. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 110–122. IEEE, 2022. 3

[CJMS23] Marcelo Campos, Matthew Jenssen, Marcus Michelen, and Julian Sahasrabudhe. A new lower bound for sphere packing. *arXiv preprint arXiv:2312.10026*, 2023. 3

[HST26] Dongming Merrick Hua, Antoine Song, and Stefan Tudose. On talagrand's convexity conjecture. *arXiv preprint arXiv:2605.10908*, 2026. 3

[Kla26] Boaz Klartag. Lattice packing of spheres in high dimensions using a stochastically evolving ellipsoid. *Inventiones mathematicae*, pages 1–29, 2026. 3