Local-to-global theorems for high-dimensional expansion

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Abstract

These notes are a re-exposition of the proofs of the local-to-global expansion theorem in simplicial complexes of [AL20] and the trickling-down theorem of [Opp18].

1 Basic definitions

Definition 1.1 (Simplicial complex)**.** A *d-dimensional simplicial complex* $\mathcal{S}$ is a collection of sets called *faces* along with a positive weight function $w : \mathcal{S} \to \mathbb{R}^+$ such that:

- if $U \in \mathcal{S}$, then any $U' \subseteq U$ is also in $\mathcal{S}$.
- for any $U \in \mathcal{S}$ there is a size-$d+1$ set $U' \in \mathcal{S}$ such that $U' \supseteq U$.
- for any set $|U| \leqslant d$, $w(U) = \sum_{\substack{U' : U' \supseteq U \\ |U'| = |U|+1}} w(U')$.

Remark 1.2. Perhaps somewhat confusingly, size-$k+1$ faces are called k-faces.

Remark 1.3. A weighting of the top-level (size-$d+1$) faces, trickles down and induces a weighting of the remaining faces.

Definition 1.4 (Link)**.** Given a simplicial complex $\mathcal{S}$ and a k-face U where $k \leqslant d-2$, we say the complex $\mathcal{S}_U$ defined as:

$$\{U' \setminus U : U' \supseteq U, U \in \mathcal{S}\}$$

along with weight function $w_U(U' \setminus U) := w(U')$ is a *k-link*.

Definition 1.5. The *k-skeleton* of $\mathcal{S}$, denoted $\mathcal{S}^{\leqslant k}$ is the complex obtained by taking all $\leqslant k$-faces of $\mathcal{S}$.

Remark 1.6. The 1-skeleton of a complex is the induced graph on it. An important object for us will be the 1-skeleton of a link. Given a k-face U, we will use M_U to denote the Markov transition matrix for the random walk on the induced graph on $\mathcal{S}_U$.

Definition 1.7 (Random walks)**.** We will be concerned with the *up-down walk* on k-faces for $0 \leqslant k \leqslant d-1$ and the *down-up walk* on k-faces for $0 \leqslant k \leqslant d$. The transition in one step of the up-down walk starting at a face U is the following:

- Choose a $k+1$-face U' containing U with probability proportional to $w(U')$.

- Drop a uniformly random element e in U' and walk to $U'' := U' \setminus \{e\}$.

One step of the down-up walk is:

- Drop a uniformly random element e in U and let $U' := U \setminus \{e\}$.
- Walk to a k-faces U'' containing U' with probability proportional to $w(U'')$.

We use P^{Δ} and P^{∇} to denote the transition matrices of the up-down and down-up walks respectively, and L^{Δ}, L^{∇} for the Laplacian operators where $L = \mathbb{1} - M$ is the Laplacian for a Markov operator M. We use $\gamma(M)$ to denote the *spectral gap* of a Markov transition matrix M. We will use the subscript k to indicate we are working with k-faces.

Remark 1.8. The up-down and down-up walks can be verified to be reversible Markov chains, and it can be verified that the stationary distribution is proportional to the weight function w on k-faces.

Remark 1.9. It can be verified that $\gamma(P_k^{\Delta}) = \gamma(P_{k+1}^{\nabla})$.

Definition 1.10. Given a Markov transition matrix M, we say $\{a, b\} \sim M$ to denote choosing a random edge by first picking a according to the stationary distribution of M and then walking to a random edge $\{a, b\}$ according to M.

Remark 1.11. When M is the Markov transition matrix of a reversible Markov chain with stationary distribution π, its spectral gap is equal to:

$$\min_f \frac{\mathbf{E}_{\{u,v\}\sim M}\left[(f_u - f_v)^2\right]}{\mathbf{E}_{u,v\sim\pi}\left[(f_u - f_v)^2\right]} = \min_f \frac{2\langle f, Lf\rangle_\pi}{\mathbf{E}_{u,v\sim\pi}\left[(f_u - f_v)^2\right]}$$

where L is the Markov Laplacian $\mathbb{1} - M$.

2 Local-to-global expansion

A useful tool in analyzing the spectral gap of the up-down walk is the following "local-to-global" theorem of [AL20].

Theorem 2.1. *Given a d-dimensional simplicial complex $\mathcal{S}$, define γ_j as $\min_{U\in j\text{-links}} \gamma(M_U)$. Then for $0 \leqslant k \leqslant d-1$:*

$$\gamma(P_k^{\Delta}) = \gamma(P_{k+1}^{\nabla}) \geqslant \frac{1}{k+2}\prod_{j=-1}^{k-1}\gamma_j.$$

Proof. We proceed by induction. When $k = 0$, the statement is clear. Suppose we know the above bound on $\gamma(P_j^{\Delta})$ for all $j \leqslant k-1$. Denoting π_j as the stationary distribution on j-faces, for any function f on the k-faces of $\mathcal{S}$:

$$\begin{aligned}\langle f, L_k^{\Delta} f\rangle_{\pi_k} &= \mathop{\mathbf{E}}_{\{U',U\}\sim P_k^{\Delta}}\left[\frac{(f_U - f_{U'})^2}{2}\right] \\ &= \mathop{\mathbf{E}}_{S\sim\pi_{k-1}}\ \mathop{\mathbf{E}}_{\{u,v\}\sim(\frac{k+1}{k+2})M_S + \frac{1}{k+2}\mathbb{1}}\left[\frac{(f_{S\cup\{u\}} - f_{S\cup\{v\}})^2}{2}\right]\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{k+1}{k+2}\gamma_{k-1} \mathop{\mathbf{E}}_{S\sim\pi_{k-1}} \mathop{\mathbf{E}}_{u,v\sim\pi_S}\left[\frac{(f_{S\cup\{u\}}-f_{S\cup\{v\}})^2}{2}\right] \\
&= \frac{k+1}{k+2}\gamma_{k-1} \mathop{\mathbf{E}}_{\{U,U'\}\sim P_k^{\nabla}}\left[\frac{(f_U-f_{U'})^2}{2}\right] \\
&= \frac{k+1}{k+2}\gamma_{k-1}\langle f, L_k^{\nabla} f\rangle_{\pi_k} \\
&\geqslant \frac{k+1}{k+2}\gamma_{k-1}\frac{1}{k+1}\prod_{j=-1}^{k-2}\gamma_j && \text{(by induction hypothesis)} \\
&= \frac{1}{k+2}\prod_{j=-1}^{k-1}\gamma_j
\end{aligned}
$$

which completes the proof. □

3 Trickling-down theorem

Sometimes, to lower bound the spectral gaps of all links in a complex it suffices to lower bound the spectral gap of only the top-level links and verify that the rest of the links are merely connected. This is articulated by the following statement due to [Opp18].

Theorem 3.1. *Given a d-dimensional simplicial complex $\mathcal{S}$, $k \leqslant d-2$, and a lower bound γ on the spectral gap of all k-links. Then for every $(k-1)$-link U, either $\gamma(M_U) = 0$ or $\gamma(M_U) \geqslant 2 - \frac{1}{\gamma}$.*

Proof. It suffices to prove the statement for $k = 0$. In particular, we assume that the link of every vertex has spectral gap at least γ and show that this implies that the graph underlying $\mathcal{S}$ either has spectral gap at least $2 - \frac{1}{\gamma}$ or is disconnected.

We do so via the following chain of inequalities:

$$
\begin{aligned}
\langle f, L_0^{\Delta} f\rangle_{\pi_0} &= \mathop{\mathbf{E}}_{\{v,w\}\sim L_0^{\Delta}}\left[\frac{(f_v-f_w)^2}{2}\right] \\
&= \mathop{\mathbf{E}}_{u\sim\pi_0}\mathop{\mathbf{E}}_{\{v,w\}\sim M_u}\left[\frac{(f_v-f_w)^2}{2}\right] \\
&\geqslant \gamma \mathop{\mathbf{E}}_{u\sim\pi_0}\mathop{\mathbf{E}}_{v,w\sim\pi_u}\left[\frac{(f_v-f_w)^2}{2}\right] \\
&= \gamma\langle f, (\mathbb{1} - (P_0^{\Delta})^2) f\rangle && \text{(by time reversibility).}
\end{aligned}
$$

Suppose L_0^{Δ} has spectral gap α, then the spectral gap of $\mathbb{1} - (P_0^{\Delta})^2$ is $1-(1-\alpha)^2 = 2\alpha - \alpha^2$, and consequently the above is at least:

$$
\alpha\gamma(2-\alpha)\mathop{\mathbf{E}}_{v,w\sim\pi_0}\left[\frac{(f_v-f_w)^2}{2}\right].
$$

Let f^* be a nonconstant vector that achieves the spectral gap of L_0^{Δ}. Then:

$$
\alpha \mathop{\mathbf{E}}_{v,w\sim\pi_0}\left[\frac{(f_v^*-f_w^*)^2}{2}\right] \geqslant \alpha\gamma(2-\alpha)\mathop{\mathbf{E}}_{v,w\sim\pi_0}\left[\frac{(f_v^*-f_w^*)^2}{2}\right]
$$

and consequently

$$\alpha \geqslant \alpha\gamma(2-\alpha).$$

Since we know $\alpha \geqslant 0$, to satisfy the above inequality either $\alpha = 0$ or $\alpha \geqslant 2 - \frac{1}{\gamma}$. $\square$

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References

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