Algorithms for Noisy Broadcast Under Erasures

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Joint work with

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MIT

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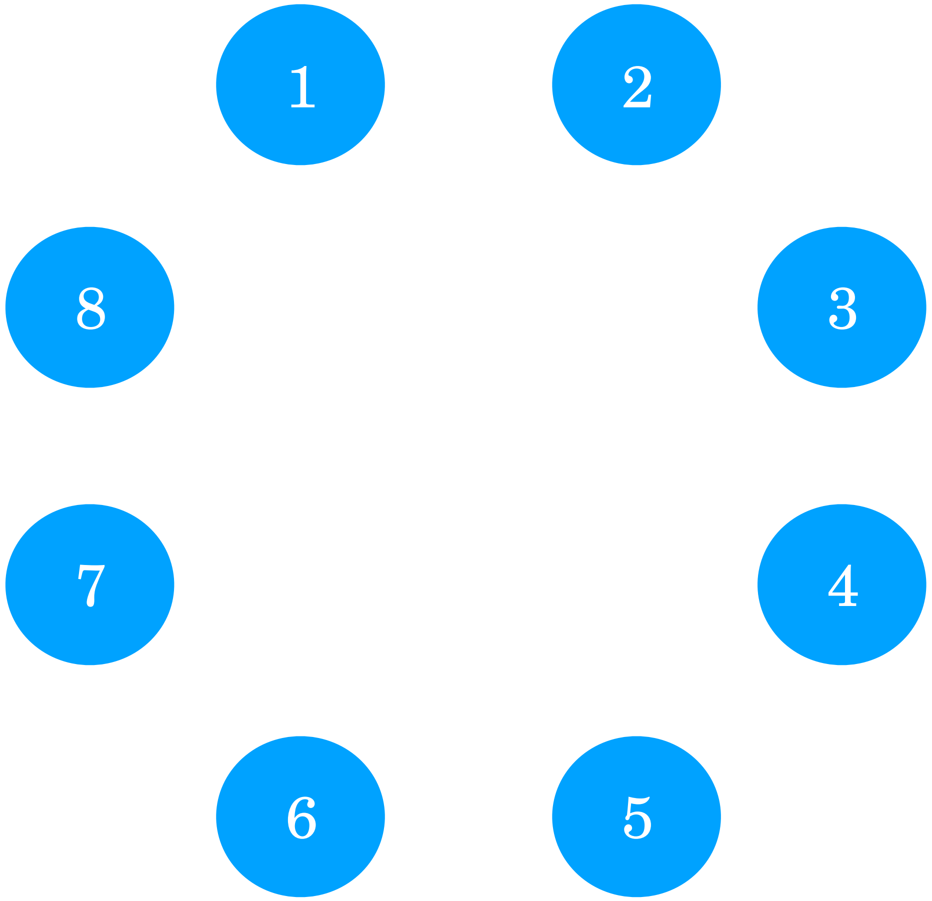
Carnegie Mellon University

n processors



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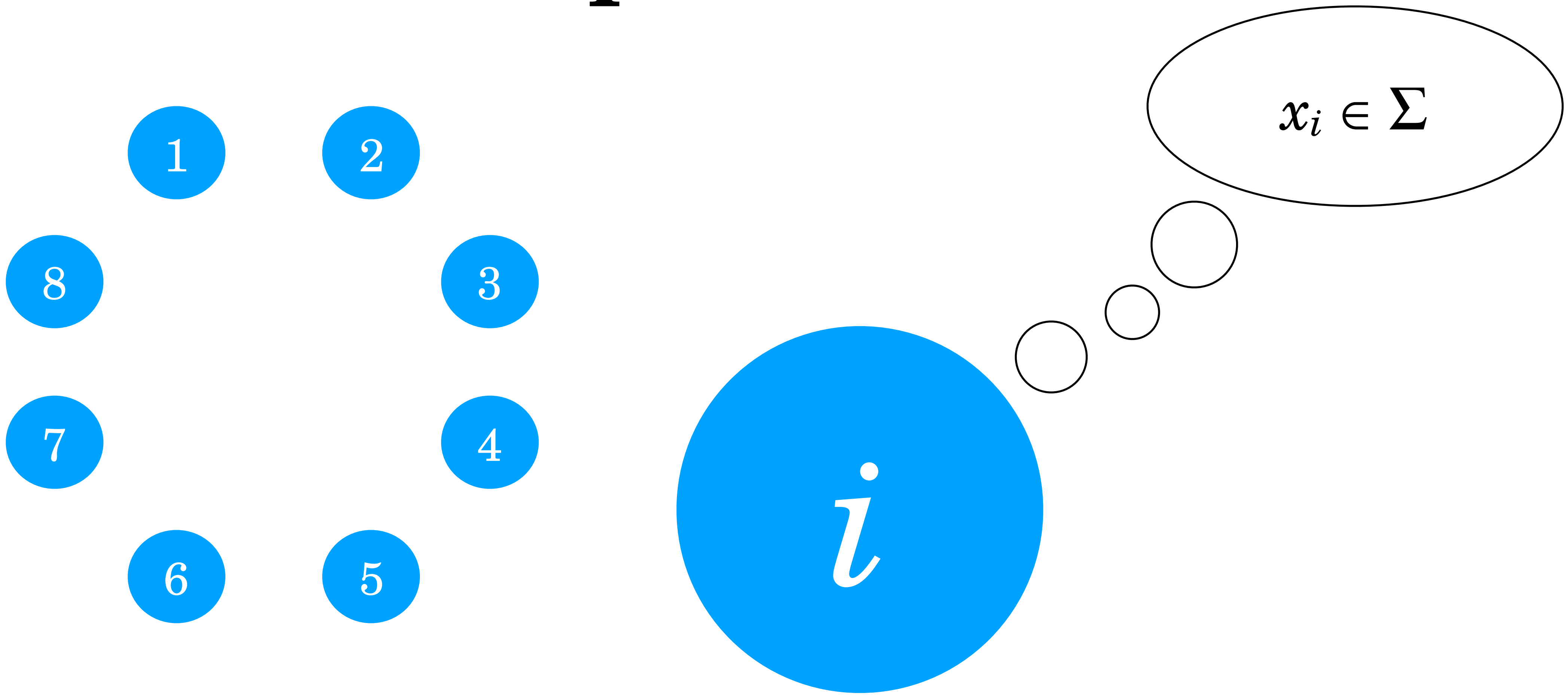

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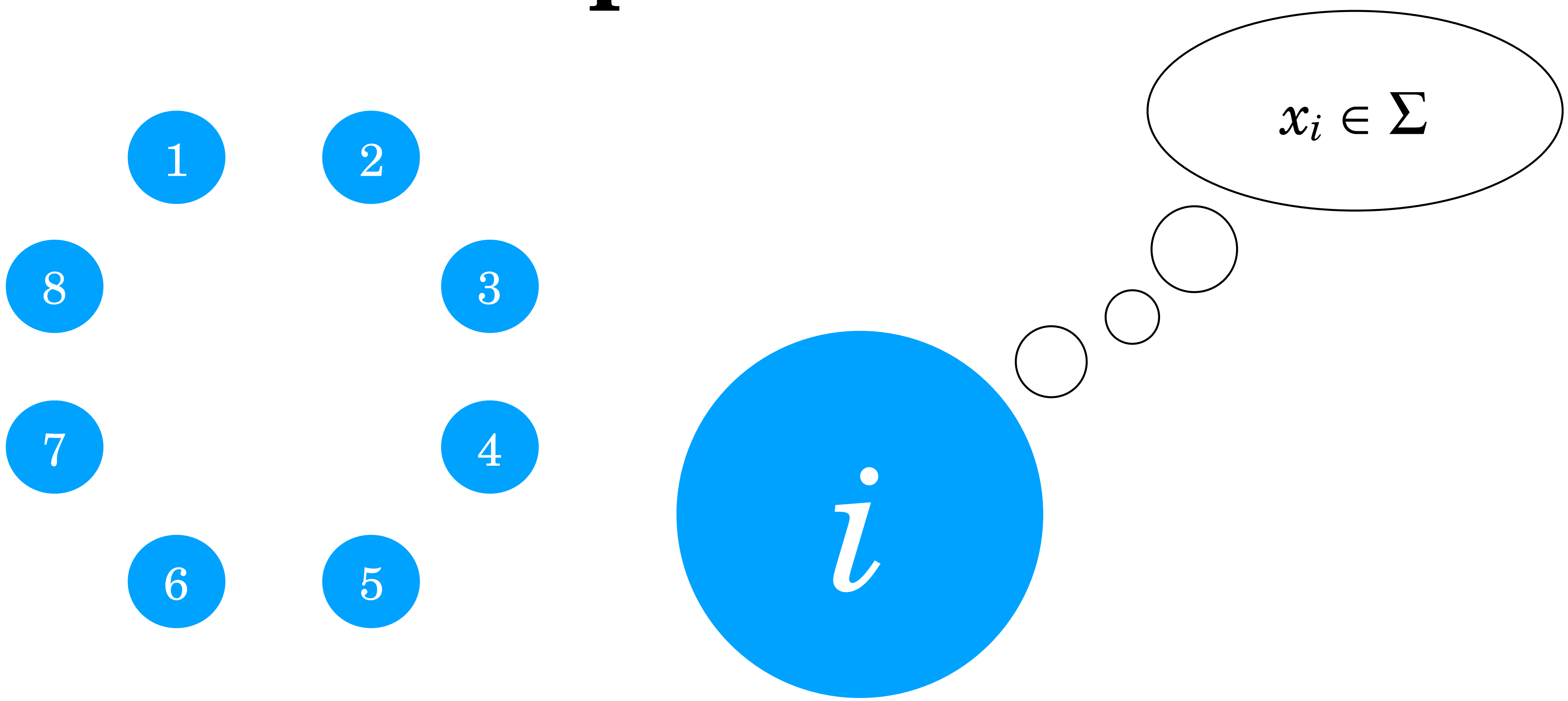

Goal: processors 1, ..., n should learn $x_1 x_2 \dots x_n$ with high probability

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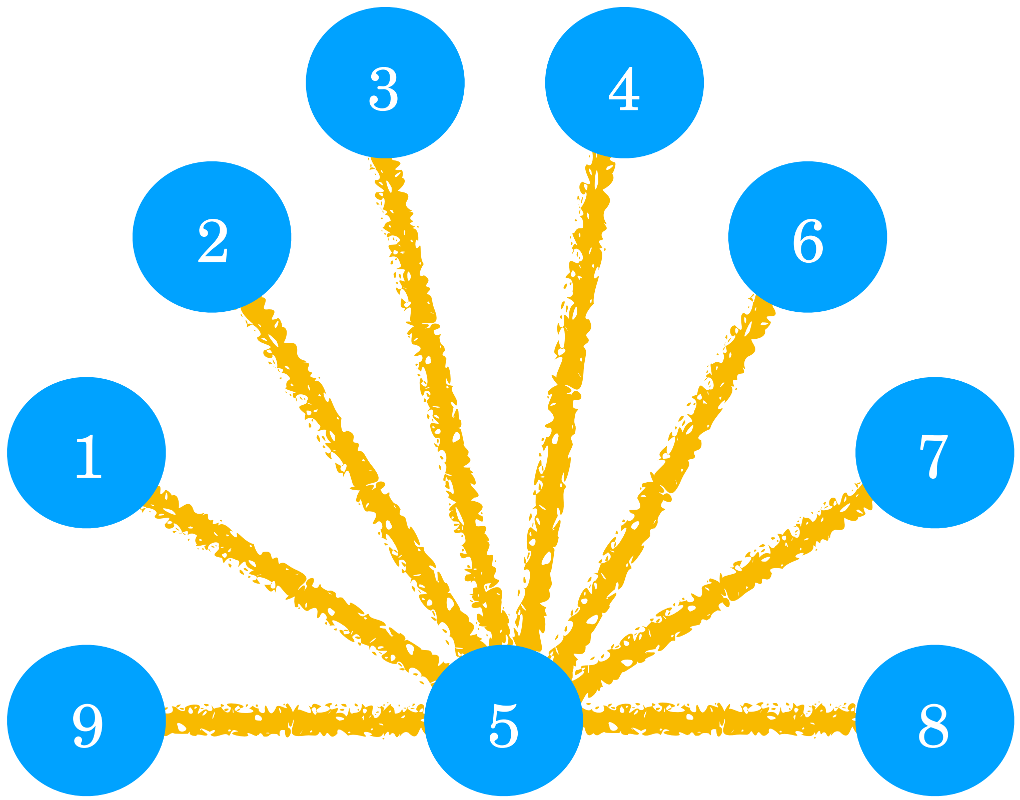
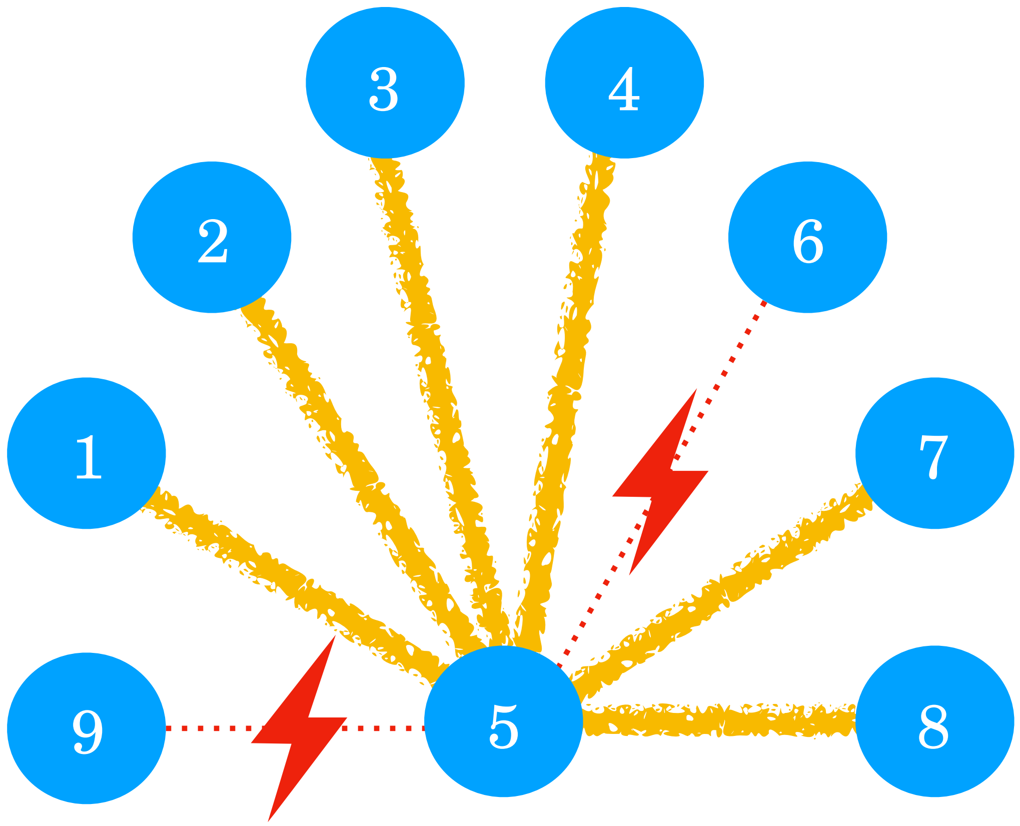
Noisy Broadcast Model



View from processor 5

- Each processor i for $1 \leq i \leq n$ broadcasts a character in Σ to all j

Noisy Broadcast Model



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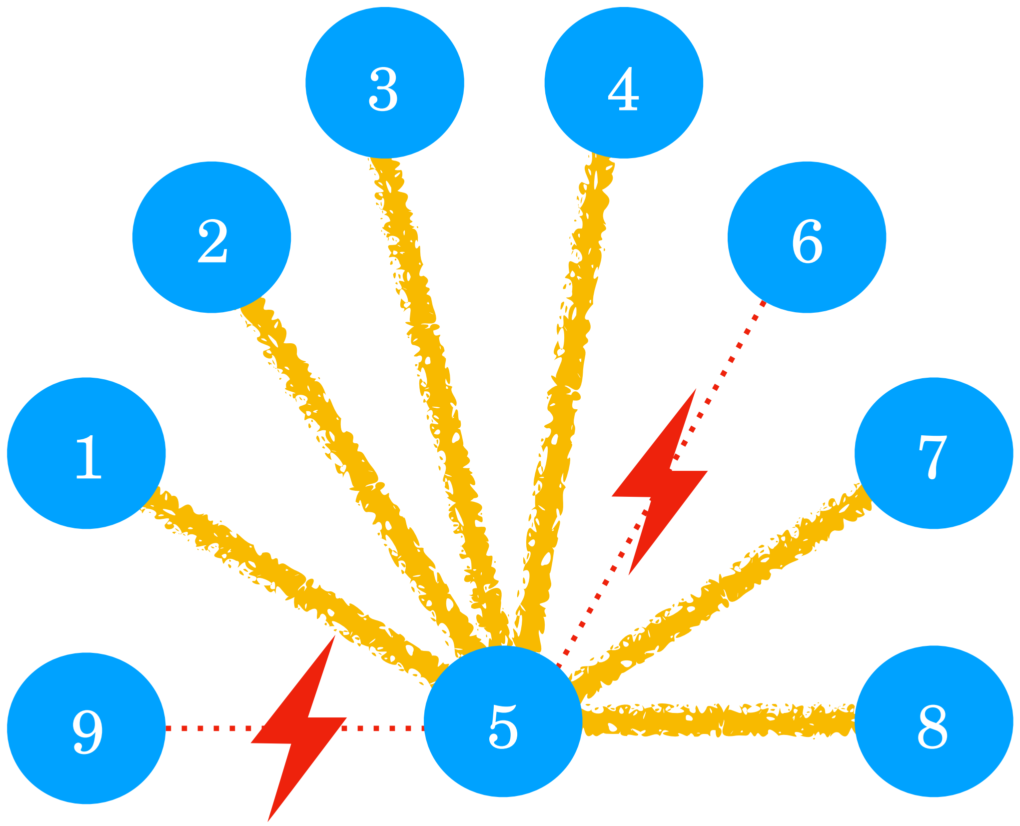

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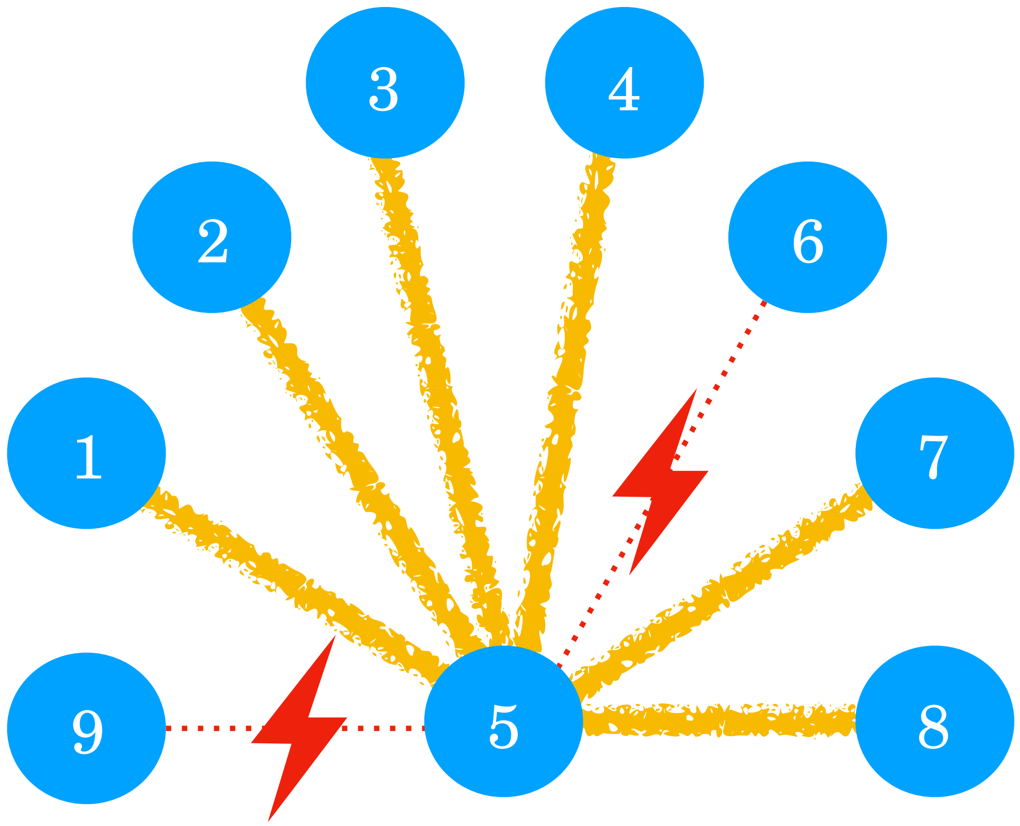

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Erasure means transmission is replaced with ***?*** at random

Note: less harsh than previously studied ***substitutions*** where transmission is replaced with a random character from Σ.

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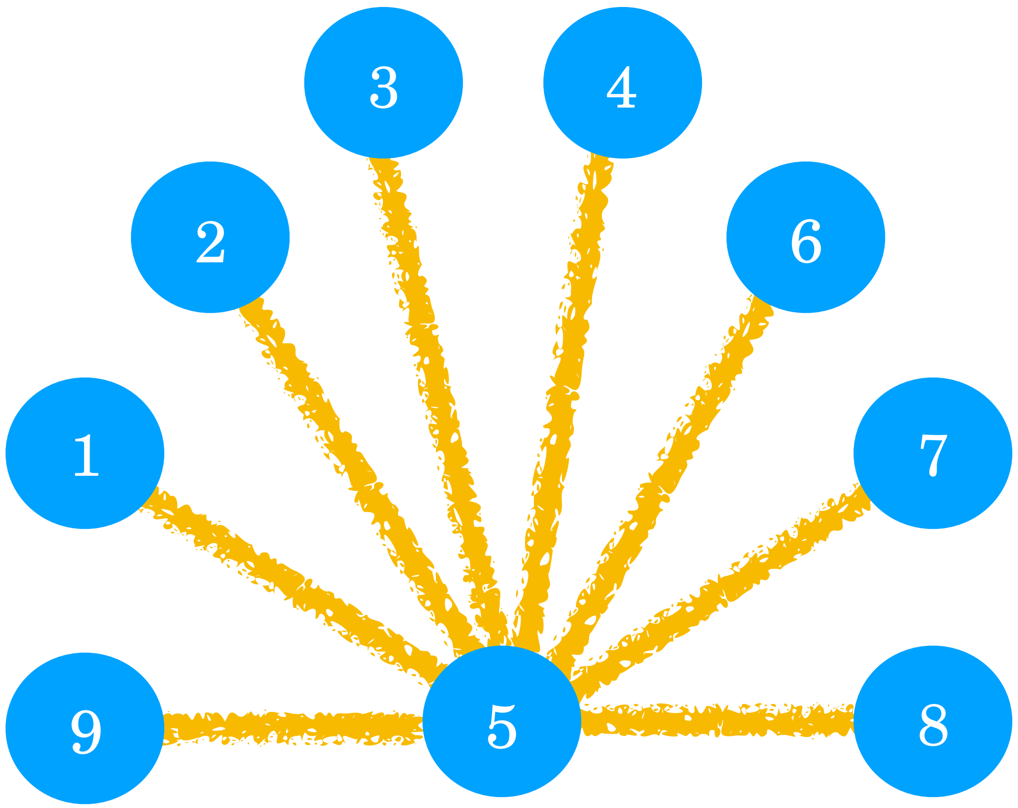
Trivial Attempt



- Processor i repeatedly broadcasts x_i for $2\log n$ rounds

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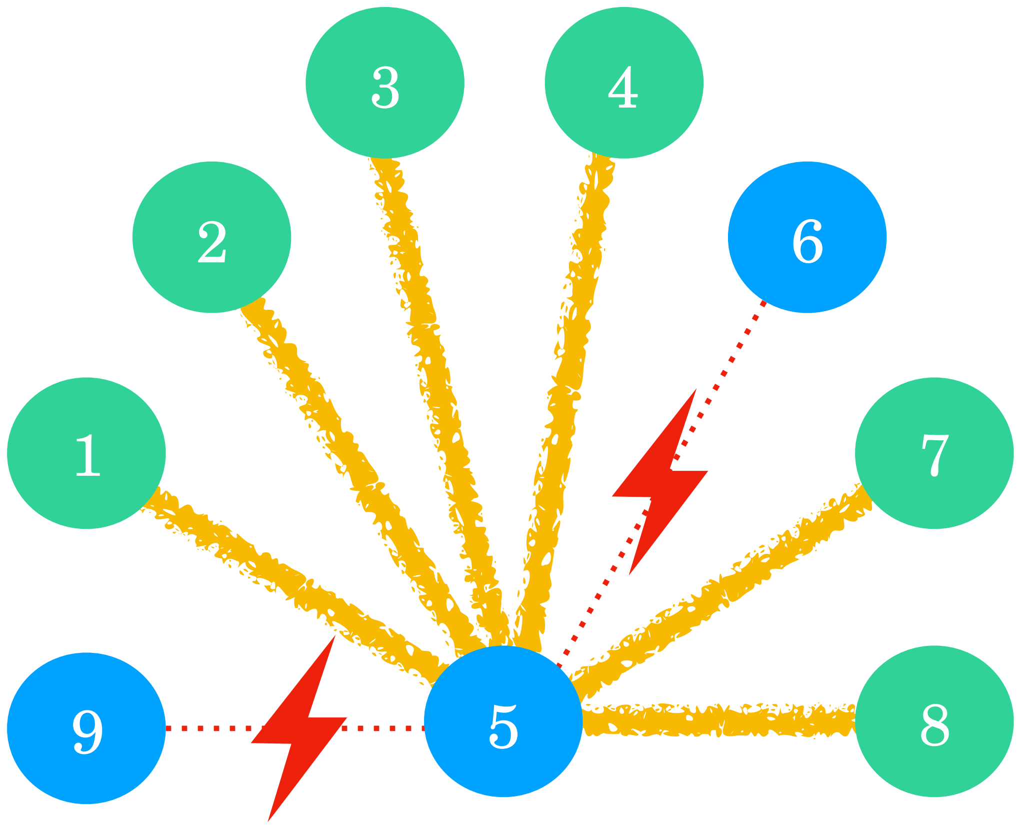
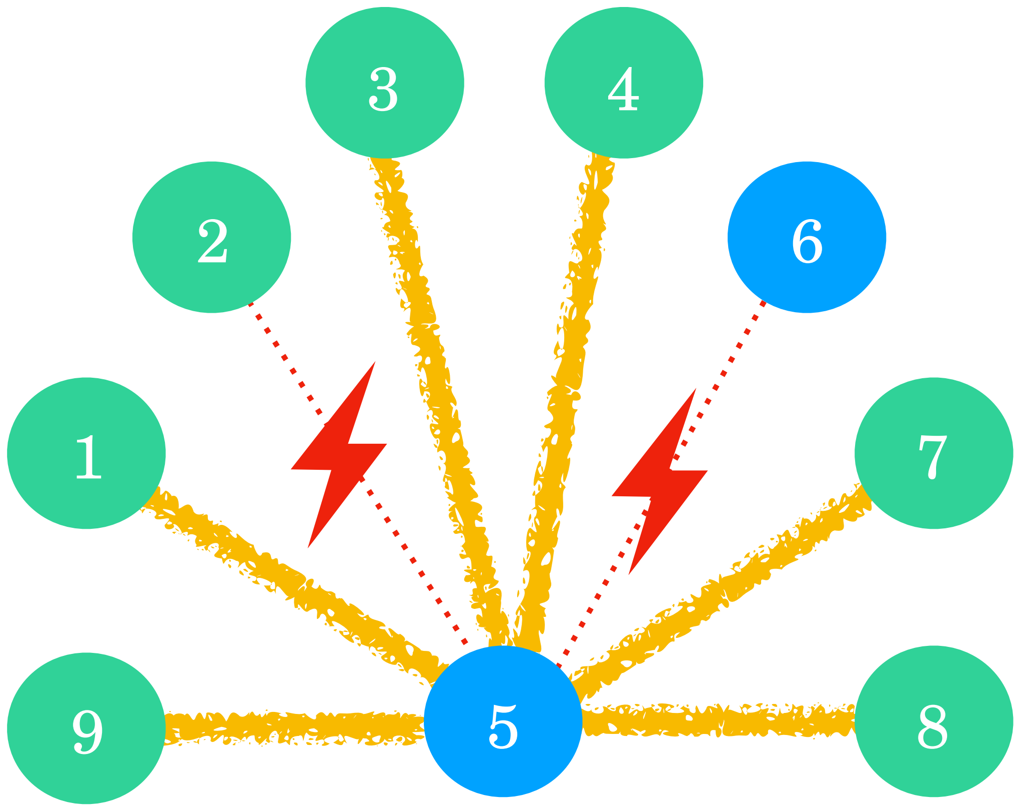


Round 1

- Processor i repeatedly broadcasts x_i for $2 \log n$ rounds

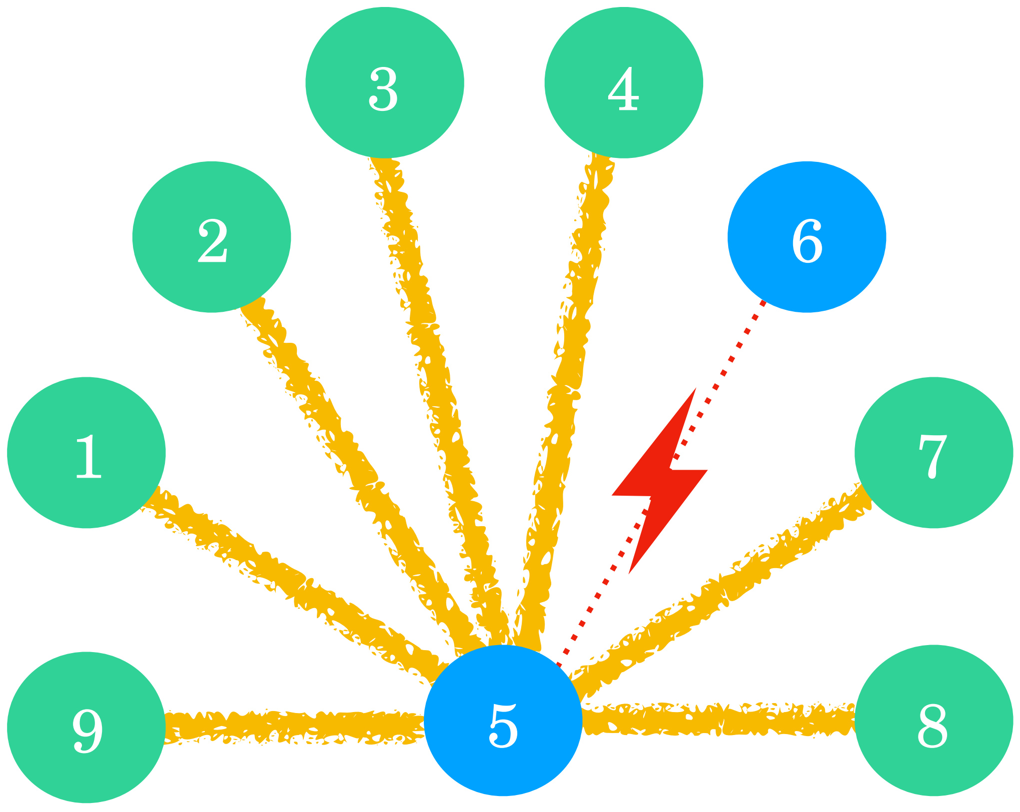
Trivial Attempt



Round 2

- Processor i repeatedly broadcasts x_i for $2\log n$ rounds

Trivial Attempt



Round 3

- Processor i repeatedly broadcasts x_i for $2\log n$ rounds

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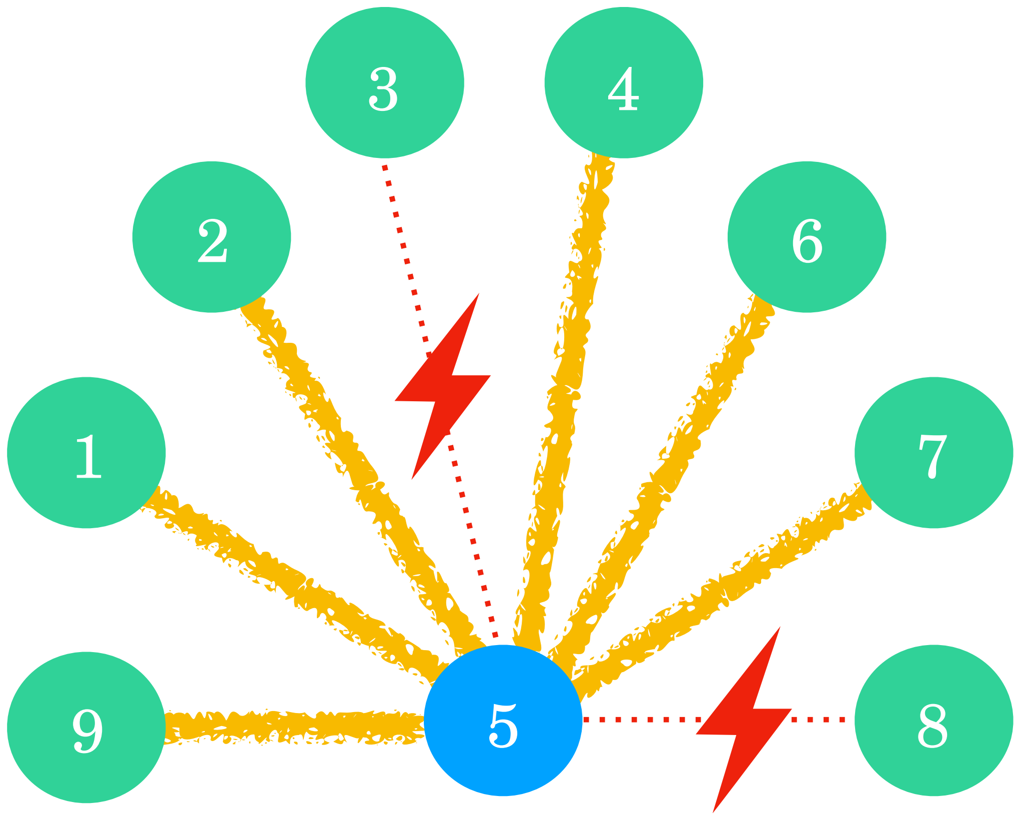
Trivial Attempt



Round 4

- Processor i repeatedly broadcasts x_i for $2\log n$ rounds

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Can we beat $O(\log n)$?

- Yes! [Gallagher'88] shows $O(\log\log n)$ rounds, even under substitutions

Can we beat $O(\log\log n)$?

- Under ***substitutions***, no! [Goyal, Kindler, Saks'08] shows $\Omega(\log\log n)$ rounds, even under substitutions
- Under ***erasures***, yes! ***[this work]***

Main Result

Recap goal: processors 1, ..., n should learn $x_1 x_2 \ldots x_n$ with high probability

$O(\log^* n)$ round algorithm for goal

$O(1)$ round algorithm when $|\Sigma| = \Omega(\text{poly}(n))$

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$O(\log^* n)$ round algorithm for goal

Focus of this talk!

$O(1)$ round algorithm when $|\Sigma| = \Omega(\mathrm{poly}(n))$

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Recap goal: processors 1, ..., n should learn $x_1 x_2 \ldots x_n$ with high probability

$O(\log^* n)$ round algorithm guarantee:

- **SUCCESS** All processors output $x_1 \ldots x_n$ with probability $\geq 1 - n^{-5}$
- **FAILURE WITHOUT KNOWLEDGE** Some processor outputs string $X \neq x_1 \ldots x_n$ with probability $\leq 2^{-7n}$
- **FAILURE WITH KNOWLEDGE** With remaining probability all processors output *'failed'*

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Failure without knowledge **is catastrophic**

Main Algorithm Outline

Recap goal: processors 1, ..., n should learn $x_1 x_2 \ldots x_n$ with high probability

$O(\log^* n)$ round algorithm outline:

A. **Learning Phase:** success probability of $1 - n^{-5}$ of all processors to learn string

B. **Validation Phase:** if A. failed, probability $1 - 2^{-7n}$ for all processors to detect failure

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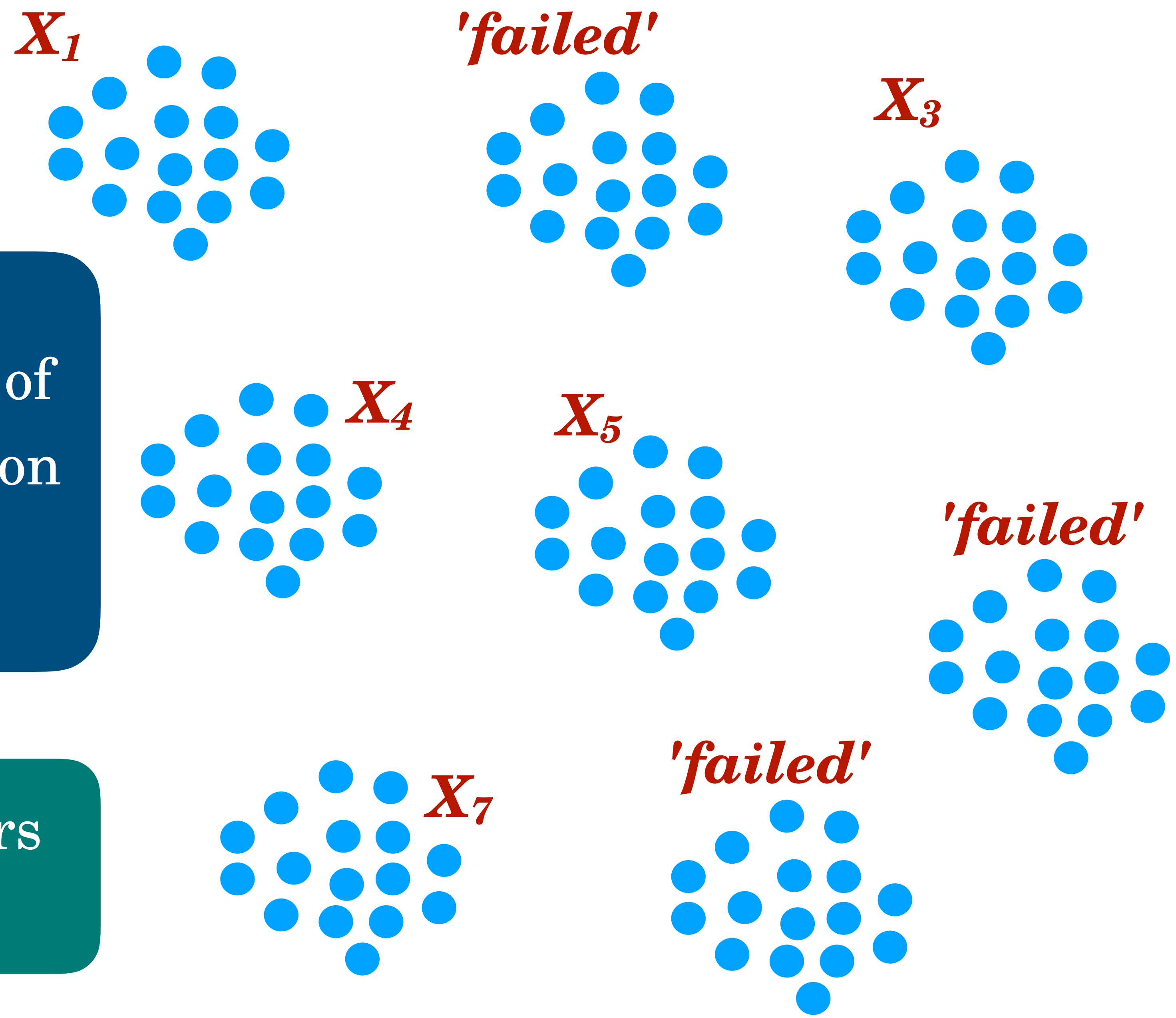
Learning Phase

Base case:

If $n \leq 10$, each processor repeatedly broadcasts 10 times

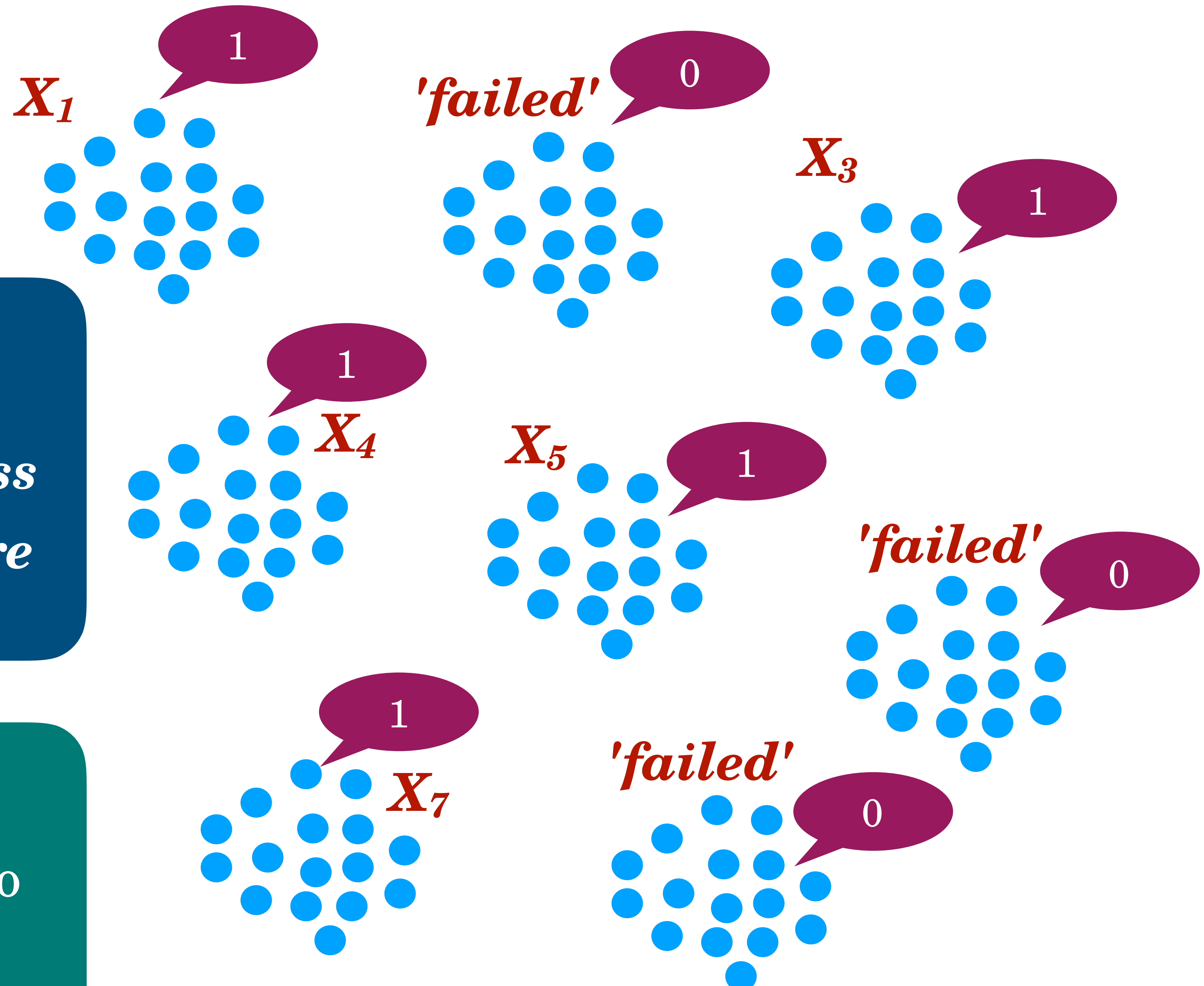
Learning Phase: Recursion



Divide into $n / \log n$ groups of size $\log n$ each and recurse on each group

Roughly $n / \log^5 n$ processors part of failed group

Learning Phase: learning failures



Processor i transmits

- 1 if recursion was ***success***
- 0 if recursion was ***failure***

Only need to receive one transmission from group to know if they failed or succeeded!

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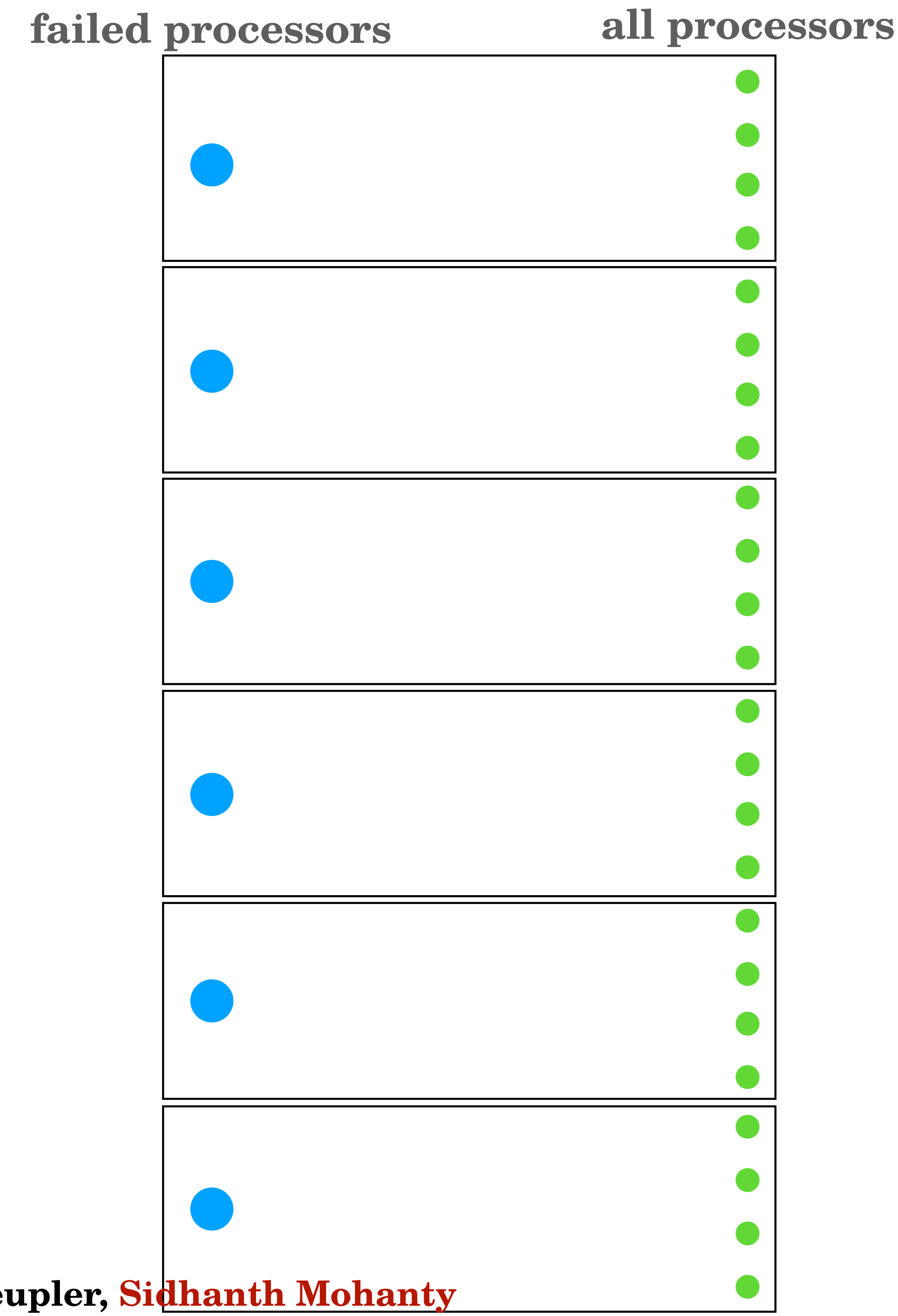
Learning Phase: correcting failures

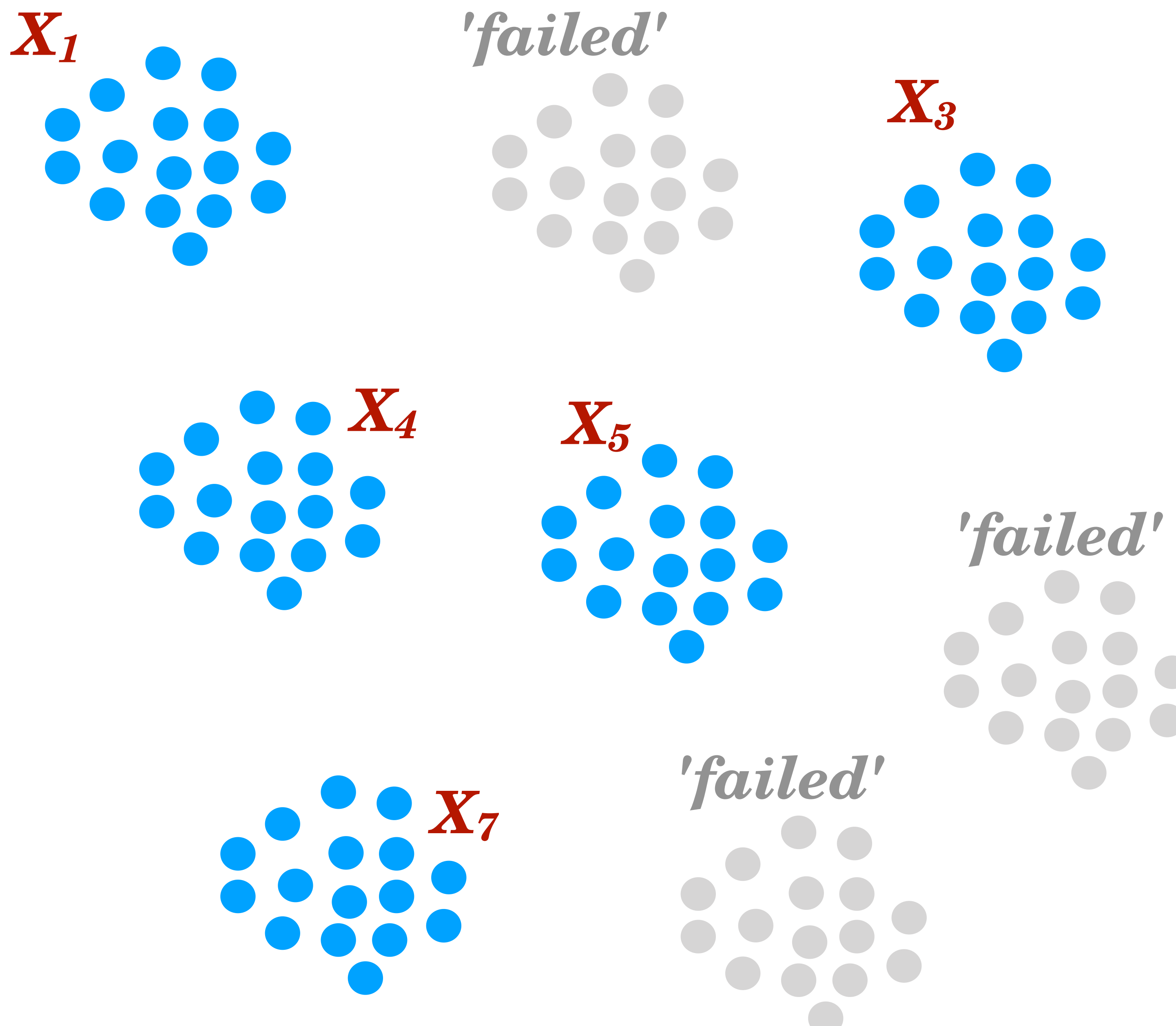
$i \log^2 n + 1$ to $(i+1) \log^2 n$
are ***responsible*** for
the i-th failed processor

- Each failed processor x broadcasts its input bit
- Each processor *responsible* for x broadcasts bit received by x

Processor i can infer input of x if it receives at least one transmission from processor responsible for x

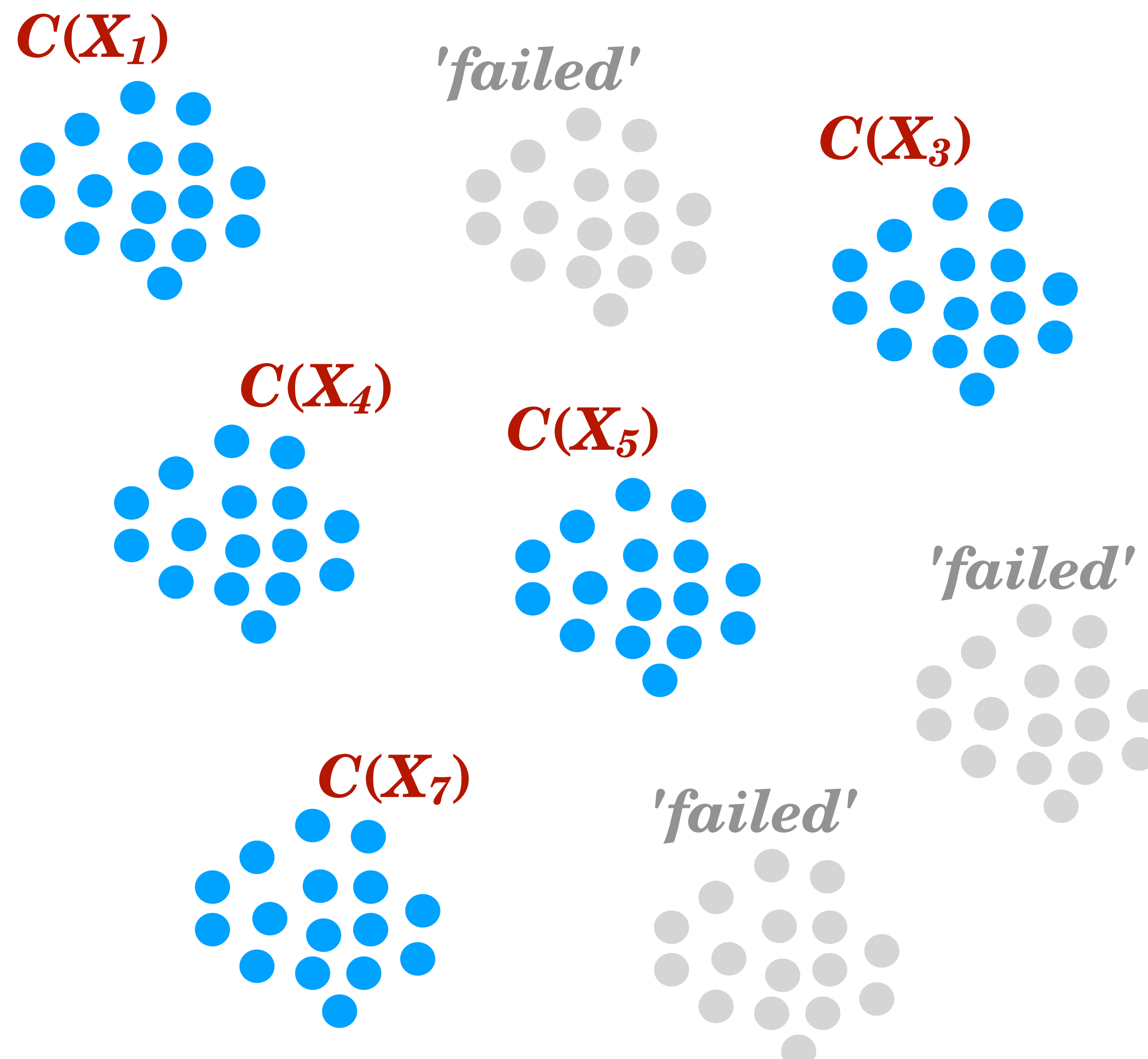


Learning Phase: success amplification



Learning Phase: success amplification

$\boldsymbol{C}$: code with decoding radius 0.25 and rate constant $\boldsymbol{K}$



Processor from group t broadcasts constant sized chunk of $\boldsymbol{C(X_t)}$ in next $1/\boldsymbol{K}$ rounds

$$X_i = a_1 \ldots a_n$$

$$\Big\downarrow C$$

$$C(X_i) = L_1 \ldots L_n$$

Each L_j has length $1/K$

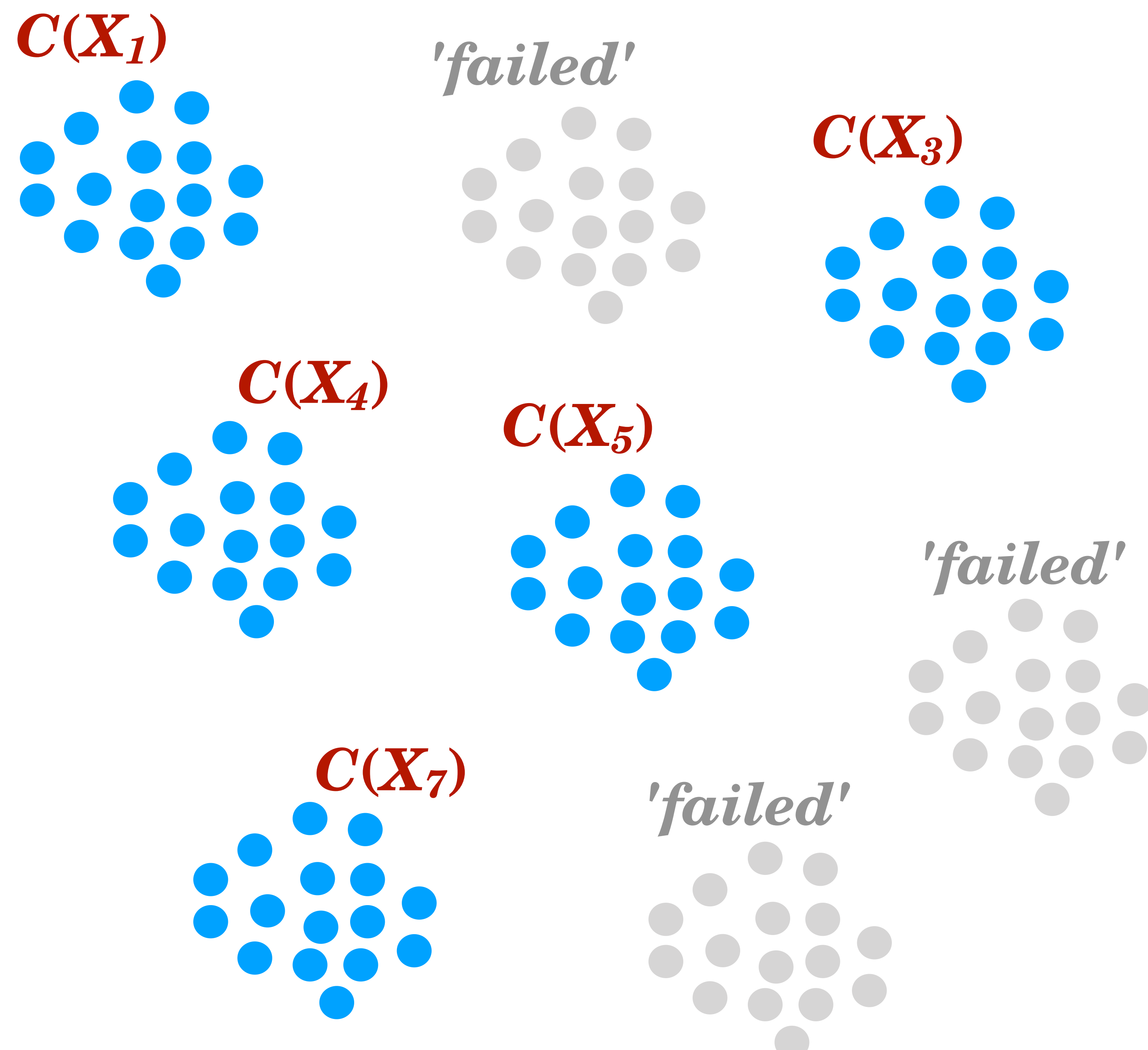
Broadcast L_j in next $1/K$ rounds

Learning Phase: success amplification

C: code with decoding radius 0.25 and rate constant K

Processor from group t broadcasts constant sized chunk of $C(X_t)$ in next $1/K$ rounds

Processor i can reconstruct X_t as long as it receives enough bits from group t



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Technical Ingredients

Ingredient 1: Algorithm to compute the **AND** of $x_1 \ldots x_n$ in $O(1)$ rounds with $\leq 2^{-\Omega(n)}$ failure probability

Processor i has string $\boldsymbol{X_i}$ as input

Ingredient 2: Algorithm to check if $\boldsymbol{X_1} = \boldsymbol{X_2} = \ldots = \boldsymbol{X_n}$ in $O(1)$ rounds with $\leq 2^{-\Omega(n)}$ failure probability

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Each processor i executes:

1. Broadcast x_i
2. Broadcast the **AND** of all received bits from round 1
3. Output the **AND** of all received bits from round 2

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$\boldsymbol{C}$: code with decoding radius 0.25 and rate constant $\boldsymbol{K}$

Outline of processor *i* execution:

1. Encode $\boldsymbol{X_i}$ with $\boldsymbol{C}$ and split into chunks
2. Broadcast assigned chunk
3. Decode received string
4. Use **ingredient 1** to check if decoded string matches $\boldsymbol{X_i}$

$$X_i = a_1 \dots a_n$$

$$C \Big\downarrow$$

$$C(X_i) = L_1 \dots L_n$$

Each L_j has length $1/K$

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Processor $\boldsymbol{i}$ receives $\boldsymbol{n} / \boldsymbol{K}$ bit string

$$\boldsymbol{S} = \boldsymbol{S_1} \ldots \boldsymbol{S_n}$$

with each $\boldsymbol{S_j}$ length $1 / \boldsymbol{K}$

Obtain codeword $\boldsymbol{T}$ within decoding radius of $\boldsymbol{S}$ and let $\boldsymbol{Y_i}$ be $\boldsymbol{C^{-1}(T)}$, the 'decoded string'

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Conclusions

- $\Omega(\log n)$ lower bound in ***substitution model*** on computing **AND** of n bits with probability $\exp(-\Omega(n))$ [Goyal-Kindler-Saks'08]
- Same problem has $O(1)$ complexity in ***erasures model*** and makes $O(\log^* n)$ algorithm possible
- Can one show an $\Omega(\log^* n)$ lower bound?