Explicit Two-Sided Vertex Expanders Beyond the Spectral Barrier

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Abstract

We construct the first explicit two-sided vertex expanders that bypass the spectral barrier.

Previously, the strongest known explicit vertex expanders were given by d-regular Ramanujan graphs, whose spectral properties imply that every small subset of vertices S has at least $0.5d|S|$ distinct neighbors. However, it is possible to construct Ramanujan graphs containing a small set S with no more than $0.5d|S|$ neighbors. In fact, no explicit construction was known to break the $0.5d$-barrier.

In this work, we give an explicit construction of an infinite family of d-regular graphs (for large enough d) where every small set expands by a factor of $\approx 0.6d$. More generally, for large enough d_1, d_2, we give an infinite family of (d_1, d_2)-biregular graphs where small sets on the left expand by a factor of $\approx 0.6d_1$, and small sets on the right expand by a factor of $\approx 0.6d_2$. In fact, our construction satisfies an even stronger property: small sets on the left and right have *unique-neighbor expansion* $0.6d_1$ and $0.6d_2$ respectively.

Our construction follows the *tripartite line product* framework of [HMMP24], and instantiates it using the face-vertex incidence of the 4-dimensional Ramanujan clique complex as its base component. As a key part of our analysis, we derive new bounds on the *triangle density* of small sets in the Ramanujan clique complex.

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Contents

1 Introduction

In this work, we study the problem of constructing explicit *vertex expanders*. Vertex expansion refers to the property that every "small enough" set of vertices should have "many" distinct neighbors. Henceforth, we restrict our attention to bipartite graphs. For $d_L > d_R$, we say that a (d_L, d_R)-biregular graph G on (L, R) is a γ*-one-sided vertex expander* if every subset $S \subseteq L$ of size at most $\eta|L|$ has at least $\gamma d_L|S|$ distinct neighbors in R for some small constant $\eta > 0$. We say that G is a γ*-two-sided vertex expander* if, additionally, every small subset S of the right vertices has at least $\gamma d_R|S|$ neighbors. When G achieves the golden standard of $\gamma \approx 1$, in we say that it is a *lossless vertex expander*.

A key motivation to study vertex expanders is for the construction of good error-correcting codes. The seminal work of Sipser and Spielman [SS96] showed that from any one-sided lossless expander, one can construct a good binary error-correcting code with a linear time decoding algorithm. In the quantum setting, the work of Lin & Hsieh [LH22] showed that *two-sided lossless expanders* with appropriate algebraic structure can be used to construct good quantum low density parity check codes.

The above applications actually go through a weaker property than lossless expansion. Sufficiently strong vertex expansion implies *unique-neighbor expansion*, the condition that every small set S has many *unique-neighbors*, or vertices with exactly one edge to S. We say G is a γ*-one-sided unique-neighbor expander* if every subset S of L of size at most $\eta|L|$ has at least $\gamma d_L|S|$ unique-neighbors in R, and that G is a γ*-two-sided unique neighbor expander* if additionally every subset S of R of size at most $\eta|R|$ has at least $\gamma d_R|S|$ unique-neighbors in L. Indeed, the above works show that any graph with $> \frac{1}{2}$-one-sided unique-neighbor expansion yields good classical codes, and $> \frac{5}{6}$-two-sided unique-neighbor expansion (and an algebraic property) yields good quantum LDPC codes.

In this work, we focus on the setting of two-sided vertex expanders, and specifically the task of constructing such objects.

Where are the vertex expanders? There are a plethora of constructions of spectral expanders, so it is natural to wonder whether one can obtain vertex expanders from them. Kahale [Kah95] proved that a Ramanujan graph, i.e., a graph with optimal spectral expansion, is a $\frac{1}{2}$-two-sided vertex expander, and demonstrated a near-Ramanujan graph on which this is tight. Unfortunately, $\frac{1}{2}$-two-sided vertex expansion falls just short of giving any unique-neighbors: in fact Kamber & Kaufman [KK22] demonstrated that the algebraic Ramanujan graph construction of Morgenstern [Mor94] contains sublinear-sized sets with *zero* unique-neighbors.

On the other hand, a random biregular graph is a two-sided lossless expander with high probability (e.g. [HLW06, Theorem 4.16]). However, the work of Kunisky & Yu [KY24, Section 4.6] gives hardness evidence that there is no efficient algorithm to certify that a random graph has unique-neighbor expansion, and in particular, to certify that a random graph has $> \frac{1}{2}$-two-sided vertex expansion, suggesting that there is no simple "algorithmic handle" for strong vertex expansion, such as the eigenvalues of some simple matrix. This motivates studying constructions with more "structure".

1.1 Our results

In this work, we construct explicit $\frac{3}{5}$-two-sided vertex expanders, breaking the spectral barrier. In fact, we prove something stronger: our graphs actually have $\frac{3}{5}$-two-sided unique-neighbor expansion. They additionally have an algebraic property relevant for constructing quantum codes. While the expansion is not quite enough to instantiate the qLDPC codes of [HL22], which demand $\frac{5}{6}$-two-sided unique-neighbor expansion, we believe this is a large step in the right direction.

Theorem 1. *For any $\varepsilon > 0$ and $\beta \in (0, 1]$, there is a large enough $d(\varepsilon, \beta)$ such that for all $d_L, d_R \geqslant d(\varepsilon, \beta)$ with $\frac{d_L}{d_R} \in [\beta, \beta + \varepsilon]$, there is an explicit infinite family of $(5d_L, 5d_R)$-biregular graphs $(Z_m)_{m \geqslant 1}$ with $\left(\frac{3}{5} - \varepsilon\right)$-two-sided unique-neighbor expansion.*

Remark 1.1. Our construction Z on vertex set (L, R) can be verified to satisfy the following *algebraic property*, relevant in the context of constructing quantum codes from vertex expanders via [HL22]: *There is a group Γ of size $\Omega(|L| + |R|)$ that acts on L and R such that $gv = v$ iff g is the identity element, and $\{gu, gv\}$ is an edge iff $\{u, v\}$ is an edge in Z.*

We use the same *tripartite line product* construction as in [HMMP24], which consists of a large tripartite base graph and a constant-sized gadget graph. In [HMMP24], the base graph was constructed using explicit bipartite Ramanujan graphs, whereas we instantiate it using the face-vertex incidence graphs of the Ramanujan clique complex of [LSV05b, LSV05a]; see also the works of Ballantine [Bal00], Li [Li04], Cartwright–Solé–Żuk [CSZ03] and Sarveniazi [Sar04]. See Section 1.3 for an overview of our analysis and the improvement over [HMMP24].

In service of proving Theorem 1, we derive bounds on the triangle density of small sets in the Ramanujan complex of [LSV05b, LSV05a], which is of independent interest in the study of high-dimensional expanders. In particular, we employ the 4-dimensional Ramanujan complex in our construction, and state the triangle bounds for the 4-dimensional case below.

Lemma 1.2 (Triangle density bound in 4D Ramanujan complex, informal)**.** *Let X be any $\mathbb{F}_q$-Ramanujan complex on n vertices, and let $U \subseteq X(0)$ be a subset of vertices of size at most δn where $q^{-25/4} > \delta > 0$. The number of size-5 faces with 3 or more vertices in U is at most $O(q^{13/2} \cdot |U|)$.*

We refer the reader to Lemma 3.11 for the more general setting, and the proof is given in Section 4.

1.2 Related work

Unique-neighbor expanders were first constructed by Alon & Capalbo [AC02], who gave several constructions, one of which involves taking a *line product* of a large Ramanujan graph with the 8-vertex, 3-regular graph obtained by the union of the octagon and edges connecting diametrically opposite vertices.

Another construction given in this work was a one-sided unique-neighbor expander of aspect ratio 22/21, which was extended by recent work of Asherov & Dinur [AD23] to obtain one-sided unique-neighbor expansion for arbitrary aspect ratio. These constructions were obtained via taking the *routed product* of a large biregular Ramanujan graph and a constant size random graph. These constructions were simplified by follow-up work of Kopparty, Ron Zewi & Saraf [KRZS24].

The work of Capalbo, Reingold, Vadhan & Wigderson [CRVW02] constructed one-sided lossless expanders with large degree and arbitrary aspect ratio via a generalization of the *zig-zag product* [RVW00]. More recently, Golowich [Gol23] and Cohen, Roth & Ta-Shma [CRTS23] gave a much simpler construction and analysis of one-sided lossless expanders based on the routed product. These routed product constructions fundamentally fall short of achieving two-sided expansion for linear size sets.

In the way of explicit constructions of two-sided vertex expanders, the work of Hsieh, McKenzie, Mohanty & Paredes [HMMP24] constructs explicit γ-two-sided unique neighbor expanders for an extremely small positive constant γ. Their construction also guaranteed two-sided lossless expansion for sets of size at most $\exp(O(\sqrt{\log n}))$. This was improved to two-sided lossless expansion for polynomial sized sets by [Che24].

A parallel line of works [TSUZ01, GUV09] construct one-sided lossless expanders in the case where the left side is polynomially larger than the right. In the same unbalanced setting, the recent work of [CGRZ24] show that the one-sided lossless expanders of [KTS22] that are based on multiplicity codes [KSY14] are in fact two-sided lossless expanders. The setting of polynomial imbalance is of interest in the literature on randomness extractors, but are not known to give good quantum LDPC codes via [LH22].

1.3 Technical overview

In this section, we give a brief overview; see Section 2.2 for a more detailed overview of the analysis, once more notation and context has been set up.

Our construction follows the *tripartite line product* framework of [HMMP24]. The first ingredient is a large tripartite base graph G on vertex set $L \cup M \cup R$ (denoting left, middle, and right vertex sets), where we place a (k, D_L)-biregular graph G_L between L and M, and a (D_R, k)-biregular graph G_R between M and R. The second ingredient is a constant-sized gadget graph H, which is a (d_L, d_R)-biregular graph on vertex set $[D_1] \cup [D_2]$. The tripartite line product between G and H is the (kd_L, kd_R)-biregular graph Z on L and R obtained as follows: for each vertex in M, place a copy of H between the D_L left neighbors of v and the D_R right neighbors of v (see Definition 2.2).

Since H has constant size, we can find an H that satisfies strong expansion properties by brute force. It is thus convenient to view H as a random biregular graph. The bipartite graphs G_L and G_R of the base graph are chosen to be appropriate bipartite expanders. In [HMMP24], they are chosen to be explicit near-Ramanujan bipartite graphs, while in our case we set them to be the vertex-face incidence graphs from high-dimensional expanders (Section 3), which give us additional structure. Specifically, we choose them to be vertex-face incidence graphs of the 4D Ramanujan complex of [LSV05b, LSV05a], and in particular, G_L and G_R are $(5, D_L)$-biregular and $(D_R, 5)$-biregular respectively. See Section 2.1 for specific properties of the base graph that we need.

For a set $S \subseteq L$, we would like to lower bound the number of its unique-neighbors (in R) in the final graph Z. The analysis starts by considering $U = N_{G_L}(S) \subseteq M$. Due to the "randomness" of the gadget H (Definition 2.8), we expect that within the gadget for each $u \in U$, almost all right-neighbors are unique-neighbors, i.e., $\approx d_L \cdot \deg_S(u)$ unique-neighbors, where $\deg_S(u) = |S \cap N_{G_L}(u)|$, as long as $|S \cap N_{G_L}(u)|$ is sufficiently small.

Therefore, we need to show that a large fraction of $u \in U$ has $\deg_S(u)$ below some threshold. We call this the *left-to-middle* analysis. Akin to [HMMP24], we split U into U_ℓ (low-degree) and

U_h (high-degree) and argue that $e(S, U_h)$ is small. In our case, we show that G_L satisfies *triangle expansion* (Definition 2.4): the property that for any small $U' \subseteq M$, there are very few vertices in L with 3 or more edges to U'. Applying this to U_h shows that most vertices in S have at most 2 edges to U_h and at least $k-2$ edges to U_ℓ. Then, barring collisions in R arising from different u, we have that most vertices in S have $\approx (k-2)d_L$ unique neighbors within gadgets they are part of.

Next, we need to argue that the unique-neighbors in gadgets corresponding to different $u \in U$ do not have too many collisions in G_R. We call this the *middle-to-right* analysis. For example, suppose a vertex $r \in R$ is a unique-neighbor within gadget H_u, if r is also a neighbor within $H_{u'}$ for some other $u' \in U$, then it is *not* a unique-neighbor of S in the final graph Z. If there is a collision between $u, u' \in U$, there must be a path between $u \to r \to u'$ in G_R.

To bound the number of collisions, we construct a multigraph C on U placing an edge for each length-2 path $u \to r \to u'$ arising from a collision. Based on spectral properties of the Ramanujan complex (specifically *skeleton expansion*; see Definition 2.5 and Lemma 3.5), we can bound the number of edges inside $\underline{C}$: the *simple* version of C obtained by replacing every multiedge by a single edge. To control the number of edges in C, we need to show that "not too many" edges occur with "abnormally high" multiplicity.

To prove the statement about multiplicities, we exploit the fact that for a pair of vertices $u, u' \in M$, the set of its common neighbors is highly constrained in the graph arising from the Ramanujan complex (Definition 2.3), and crucially, the property that neighborhoods of small sets in H are "spread out": i.e. for a vertex $u \in M$, and any small subset of its left neighbors S_u, the neighborhood of S_u in the gadget graph H does not place "too many" vertices on the neighbors of any fixed u'; this is articulated in Definition 2.8.

Why does using HDX do better than [HMMP24]? [HMMP24] use near-Ramanujan bipartite graphs as the base graph. In their middle-to-right analysis, via sharp density bounds of small subgraphs in bipartite spectral expanders, they bound the number of collisions each vertex $u \in M$ partakes in by $\sqrt{D_R}$. As a result, they require the gadget degrees d_L, d_R to be larger than $\sqrt{D_L}, \sqrt{D_R}$ by at least some constant factor, which hurts unique-neighbor expansion within gadgets corresponding to $u \in M$ such that $\deg_S(u) \approx \sqrt{D_L}$.

One of the key properties in the Ramanujan complex we use is: the square of the graph G_R restricted to M looks like ℓ-copies of an almost-Ramanujan graph of degree D_R/ℓ, for some "reasonably large" ℓ. As an upshot, the number of *other vertices* $u' \in M$ that u has a collision with is at most $\sqrt{D_R/\ell}$. We can use this to show that as long as we choose our gadget degree $d_R \gg \sqrt{D_R/\ell}$, we can prove that for a typical vertex $u \in M$, only a small fraction of the unique-neighbors within its gadget encounters a collision. The win over the approach of [HMMP24] comes from the ability to choose d_R such that $\sqrt{D_R/\ell} \ll d_R \ll \sqrt{D_R}$. This improves the range of values of $\deg_S(u)$ for which the gadget corresponding to vertex u experiences lossless expansion.

Going beyond $\frac{3}{5}$. For our construction, we choose $k = 5$ which gets us an expansion factor of $\frac{k-2}{k} = \frac{3}{5}$ in Theorem 1. One might ask if we could get a better expansion factor by choosing a larger k. Unfortunately, our analysis requires us to balance certain parameters, and we were not able to show that larger k satisfies the necessary inequalities. See Remark 2.14 for a discussion on parameter choice.

We mention a few candidate ways to extend our analysis beyond $\frac{3}{5}$. First, if one could improve the bounds on triangle density in small sets, that may allow larger k to satisfy the necessary inequality, thus improving the bound to $\frac{k-2}{k}$. One could also attempt to bound *tetrahedron expansion* (or even larger faces) in place of triangle expansion, in hopes of satisfying the necessary inequalities for appropriately larger k. However, in both approaches, a key difficulty seems to be that there are very few bounds known for the incidences between planes of various dimensions, and those known are extremely weak for the size of sets that naturally arise in the links of the Grassmanian clique complex.

A concrete question in this direction is whether we can obtain tight subgraph density bounds for the bipartite incidence graph between dimension i and j subspaces of $\mathbb{P}(\mathbb{F}^k)$, for sets of size $q^{i(d-i)/2}$ and $q^{j(d-j)/2}$. As we will see in Section 4, we need such tight bounds as these graphs appear as "average" links in the Ramanujan complex.

1.4 Notation

We now establish some notational conventions we follow throughout the paper.

For an n-vertex graph G, we use A_G to denote the adjacency matrix of G, and write its eigenvalues in descending order $\lambda_1(G) \geqslant \ldots \geqslant \lambda_n(G)$. We say an *eigenvalue of* G to mean *eigenvalue of the adjacency matrix of* G. When G is bipartite on left vertex set A, right vertex set B, and edge set E, we write it as (A, B, E). We use $G^\top$ to denote the bipartite graph (B, A, E). Sometimes, when the edge set is clear, we will simply denote G by (A, B).

We mildly deviate from standard notation for convenience, and define $[k] := \{0, 1, \ldots, k-1\}$.

2 Explicit construction of $3/5$-two-sided unique-neighbor expanders

In this section, we prove our main theorem.

Theorem 2.1 (Formal Theorem 1)**.** *For any $\beta \in (0, 1]$ and $\varepsilon > 0$, there exists $d_0 \in \mathbb{N}$ such that for any $d_L, d_R \geqslant d_0$, there is an infinite family $(Z_m)_{m \geqslant 1}$ of $(5d_L, 5d_R)$-biregular bipartite graphs on (L, R) such that $d_R/d_L \in [\beta, \beta + \varepsilon]$ for which Z_m is a $(3/5 - \varepsilon)$-unique-neighbor expander. Further, there is an algorithm that takes in n as input, and in $\mathrm{poly}(n)$-time constructs some Z_m from this family for which $|V(Z_m)| = \Theta(n)$.*

In Section 2.1, we describe our construction and state a more general result (Theorem 2.11), which directly implies Theorem 2.1. In Section 2.2, we give a proof overview for Theorem 2.11, and we formally prove Theorem 2.11 in Section 2.3.

2.1 Construction

The construction is based on the "tripartite line product" in [HMMP24, Definition 7.6], instantiated with more structured base and gadget graphs.

Definition 2.2 (Construction)**.** The ingredients for our construction are:

- Two bipartite graphs $G_L = (L, M, E_L)$ and $G_R = (R, M, E_R)$, where G_L is k-regular on L and D_L-regular on M, and the graph G_R is k-regular on R, and D_R-regular on M. For each vertex v in M, we order its neighbors in L and R according to injective functions $\mathrm{LNbr}_v : [D_L] \to L$ and $\mathrm{RNbr}_v : [D_R] \to R$ respectively.

- A constant-sized *gadget graph* H, which is a bipartite graph with left vertex set $[D_L]$, and right vertex set $[D_R]$. The graph H is d_L-regular on the left, and d_R-regular on the right.

The final construction, which we call Z, is a bipartite graph on (L, R) constructed by taking each vertex $u \in M$, and placing a copy of H between the left and right neighbors of u, which we will denote as H_u. More concretely, for every $u \in M$, $i \in [D_L]$, and $j \in [D_R]$, we place an edge between $\mathrm{LNbr}_u(i)$ and $\mathrm{RNbr}_u(j)$ if there is an edge between i and j in H. We emphasize that M is only used in the construction of Z and does not appear in the final graph.

Choice of base graph. We choose G_L and G_R as (truncated) bipartite vertex-face incidence graphs of the Ramanujan complex. We state the relevant properties of the base graph we use below, and defer the proof that such a base graph indeed exists to Section 3.

Definition 2.3 (Structured bipartite graph)**.** A (k, D)-biregular *structured bipartite graph* $G = (V, M, E)$ is a bipartite graph between V and M where:

1. Every vertex in V is degree-k and every vertex in M is degree-D.

2. For each vertex $u \in M$, there is an ordering of its neighbors specified by an injective function $\mathrm{Nbr}_u : [D] \to V$.

3. The set M can be expressed as a disjoint union $\sqcup_{a \in [k]} M_a$ such that each $v \in V$ has exactly one neighbor in each M_a.

4. For each pair of distinct $a, b \in [k]$, there exists $s_G(a, b)$ (abbreviated to s) such that there are s *special sets* $(A_i \subseteq [D])_{i \in [s]}$ of size between $\frac{D}{2s}$ and $\frac{2D}{s}$, such that for any $u \in M_a$ and $v \in M_b$, we have $N(u) \cap N(v)$ is either empty, or is equal to $\mathrm{Nbr}_u(A_i)$ for some $i \in [s]$.

As a prelude to Section 3, we will construct structured bipartite graphs (V, M, E) to be the incidence graph between vertices and $(k-1)$-faces in the Ramanujan clique complex of [LSV05b, LSV05a], where we set V to be the set of $(k-1)$-faces and M to be the vertex set. Then, the properties listed in Definition 2.3 will be satisfied naturally; see Theorem 3.4.

We now describe some quantities associated to a (k, D)-biregular structured bipartite graph $G = (V, M, E)$ that are of interest in the analysis.

Definition 2.4 (Small-set triangle expansion)**.** We say that G is a *τ-small-set triangle expander* if for some small constant $\eta > 0$, depending on k and D, and every $U \subseteq M$ of size at most $\eta|M|$, the number of vertices $v \in V$ with 3 or more neighbors into U is at most $\tau \cdot |U|$.

Definition 2.5 (Small-set skeleton expansion)**.** Let $\widetilde{G}$ be the simple graph obtained by placing an edge for every $u, v \in M$ such that there is at least one length-2 walk (u, a, v) in G for $a \in V$. We say that G is a *λ-small-set skeleton expander* if for some small constant $\eta > 0$, and every set $U \subseteq M$ of size at most $\eta|M|$, the largest eigenvalue of $\widetilde{G}[U]$ is at most λ.

In the following, we define notation for our construction.

Notation 2.6 $(G_L, G_R, D_L, D_R, k, \tau, \lambda, s_L, s_R)$**.** We choose our tripartite base graph on (L, M, R) with the following two structured bipartite graphs: $G_L = (L, M, E_L)$, which is (k, D_L)-biregular, and $G_R = (R, M, E_R)$, which is (k, D_R)-biregular. Let τ and λ be constants such that both G_L and G_R are τ-small set triangle expanders, and λ-small-set skeleton expanders. We use $s_L(a, b)$ and $s_R(a, b)$ to refer to $s_{G_L}(a, b)$ and $s_{G_R}(a, b)$ respectively.

In Section 3, we prove the following about the existence of base graphs G_L and G_R. Concretely, the below statement follows from Lemma 3.18.

Lemma 2.7. *Given integers n_0, k, prime power q, integers D_L and D_R that are multiples of $k!$ and have magnitude at most $c \cdot q^{\binom{k}{2}}$ for some small constant $c > 0$ as input, there is a $\mathrm{poly}(n_0)$-time algorithm that constructs (L, M, R) where $|M| = n = \Theta(n_0)$, and $|R| = |L| \cdot D_L / D_R$, and outputs structured bipartite graphs G_L on (L, M) and G_R on (M, R) such that:*

$$D_L,\ D_R = \Theta_k(1) \cdot q^{\binom{k}{2}},$$

$$\tau = O_k(1) \cdot q^{\binom{k}{2} - \frac{k^2}{8} - \frac{1}{2}} \cdot \max_{0 \leqslant i_0 < i_1 < i_2 < k} \ \min_{(i,j) \in \left\{ \substack{(i_1 - i_0, i_2 - i_0),\\ (i_2 - i_1, k + i_0 - i_1),\\ (k + i_0 - i_2, k + i_1 - i_2)} \right\}} q^{\frac{1}{8}((i-j+2)^2 + (k-i-j)^2)},$$

$$\lambda = O_k(1) \cdot q^{\frac{1}{2}\lfloor \frac{k^2}{4} \rfloor},$$

$$s_L(a,b),\ s_R(a,b) \in \left[q^{k-1}, O(q^{\lfloor k^2/4 \rfloor})\right] \quad \forall a < b \in [k].$$

Specifically for $k = 5$, we have $D_L, D_R = \Theta(q^{10})$, $\tau = O(q^{6.5})$ (see Corollary 4.9), $\lambda = O(q^3)$, and $s_L, s_R \in [q^4, O(q^6)]$.

Choice of gadget graph. We choose H as a constant-sized lossless expanders with some pseudorandom properties, whose motivation will be clearer in the analysis. One should think of H as being a random graph; the precise properties we need are articulated in Definition 2.8.

Definition 2.8 (Pseudorandom gadget)**.** Let H be a (d_L, d_R)-biregular bipartite graph on vertex set $([D_L], [D_R])$. For each $a, b \in [k]$, let $(A_i \subseteq [D_R])_{i \in [s]}$ be the *special sets* from Definition 2.3 where $s = s_R(a,b)$. Define $D := D_L + D_R$. We say H is a *pseudorandom gadget* if for every $a, b \in [k]$, we have the following properties:

1. For every $S \subseteq [D_L]$ such that $|S| \leqslant D_R / d_L$ and for every $W \subseteq [s]$ with $|W| \geqslant \frac{s \log D}{d_L}$,
$$\sum_{i \in W} |N(S) \cap A_i| \leqslant 32|W| \cdot \max\left\{ \frac{1}{r} \cdot d_L |S|,\ \log D \right\}.$$
2. For any $S \subseteq [D_L]$ with $|S| = o_D(1) \cdot D_R / d_L$, we have $|N(S)| \geqslant (1 - o_D(1)) d_L |S|$.

Definition 2.9 (Good gadget)**.** A *good gadget* graph H is a (d_L, d_R)-biregular graph on $([D_L], [D_R])$ such that H and $H^\top$ are both pseudorandom gadgets.

Lemma 2.10. *For $d_L, d_R \geqslant \log^2 D$, and $d_R = o_D(1) \cdot D_L$, there exists a good gadget graph H.*

In Section 5, we will prove Lemma 2.10 by showing that a random gadget satisfies the desired properties with high probability.

Combining the parts. Finally, we state our main theorem below.

Theorem 2.11. *Suppose for some $\delta > 0$, we have:*

$$\frac{1}{\lambda} \leqslant \delta \leqslant o_D(1) \leqslant \frac{1}{2k}, \quad \max_{a,b \in [k]} \left\{ \lambda, \sqrt{s_L(a,b)}, \sqrt{s_R(a,b)} \right\} \frac{\log^2 D}{\delta} \leqslant d_L, d_R \leqslant \frac{\delta D}{\tau \log D},$$

$$\lambda \leqslant \delta^2 \cdot \min_{a,b\in[k]} \{s_L(a,b), s_R(a,b)\}.$$

Let Z be a (kd_L, kd_R)-biregular graph instantiated according to Definition 2.2 with a base graph satisfying the properties listed in Notation 2.6, and a good gadget graph. There is a constant $\eta > 0$ such that

- *for every subset $S \subseteq L(X)$, where $|S| \leqslant \eta|L(X)|$, S has $(1 - \delta - o_D(1)) \cdot (k-2) \cdot d_L|S|$ unique-neighbors in Z, and*
- *for every subset $S \subseteq R(X)$ where $|S| \leqslant \eta|R(X)|$, S has $(1 - \delta - o_D(1)) \cdot (k-2) \cdot d_R|S|$ unique-neighbors in Z.*

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. The statement immediately follows by instantiating Theorem 2.11 with: (1) the base graph from Lemma 2.7 with $k = 5$, q large enough so that $d_L, d_R = \Theta(q^{3.25})$, and $D_L, D_R = \Theta(q^{10})$, and (2) the gadget graph from Lemma 2.10. By Lemma 2.7, the base graph is efficiently constructible with $\tau = O(q^{6.5})$ and $\lambda = O(q^3)$ (see Corollary 4.9 for the bound on τ), and $q^4 \leqslant s_L(a,b), s_R(a,b) \leqslant O(q^6)$ for all $a, b \in [k]$. The gadget graph is also efficiently constructible via a brute force search over all graphs on $([D_1], [D_2])$. Finally, the tripartite line product can also be computed efficiently. □

2.2 Proof overview of Theorem 2.11

For Theorem 2.11, we will only prove the unique-neighbor expansion for $S \subseteq L(X)$; the argument for $S \subseteq R(X)$ is exactly the same. Similar to [HMMP24], we split the proof of Theorem 2.11 into two parts: left-to-middle and middle-to-right. Given a set $S \subseteq L(X)$, let $U \subseteq M$ be the neighbors of S in M in the base graph.

Left-to-middle analysis. We divide U into U_ℓ (low-degree) and U_h (high-degree), and we would like to prove that a large (close to $\frac{k-2}{k}$) fraction of edges leaving S go into U_ℓ. This is desirable because in our gadget H (satisfying Definition 2.8), small subsets (of $[D_1]$) expand losslessly while we have no guarantees on large subsets.

To do so, we apply the small-set triangle expansion (Definition 2.4) of the base graph G_L to U_h, which roughly states that very few vertices in S have more than 2 edges to U_h. Since each vertex in S has degree k, this implies that most vertices in S have at least $k-2$ edges going to U_ℓ, thus giving a lower bound on $e(S, U_\ell)$. This is articulated in Lemma 2.15.

Middle-to-right analysis. We need to argue that the unique-neighbors from each $u \in U$ (within the gadget H_u) do not have too many collisions. It is convenient to visualize such collisions as follows: (1) for each $u \in U$, we draw blue edges from u to all its right neighbors that are unique-neighbors of $N_L(u)$ within the gadget H_u; (2) we draw red edges from u to all its right neighbors that are nonunique-neighbors of $N_L(u)$ within H_u. The following is a simple observation:

Observation 2.12. The set of unique-neighbors of S in the final graph Z is the set of vertices in R incident to exactly one blue edge and no red edge.

We note that since vertices in U_ℓ have low degree on the left, they will have many blue edges going to R (almost $(k-2)d_L|S|$ in total). We next need to prove that few collisions occur.

Bounding collisions. We define a (multi-)graph C on vertex set U by placing a copy of the edge $\{u, v\}$ for each $u \in U_\ell$, $v \in U$, $r \in R$ such that u has a blue edge to r and v has an (either blue or red) edge to r. Then, the number of collisions is exactly $e(C)$.

A collision between $u, v \in U$ can occur only if u, v have common neighbors in R. In other words, the graph $\underline{C}$, defined as the graph obtained by removing parallel edges in C, is a subgraph of $\widetilde{G}_R$ — the simple graph on M obtained from length-2 walks in G_R (see Definition 2.5). The most natural attempt to bound $e(C)$ is to use the small-set skeleton expansion of $\widetilde{G}_R$ (Definition 2.5), which implies that small subgraphs in $\widetilde{G}_R$ have small average degree (see Lemma 2.17). However, C can have multiplicities, which complicate the analysis.

Let us examine where the multiplicities may come from. An edge $\{u, v\}$ in C can have multiplicites when they share several common neighbors in R in the bipartite graph G_R. To analyze the common neighborhood structure, we use the fact that G_R satisfies Item 4 in Definition 2.3, which states that the common neighborhood of $u \in M_a$ and $v \in M_b$ is either empty or equal to some"special set". We thus have the following crucial observation:

Observation 2.13. For $u \in M_a$ and $b \neq a$, consider the $s_{G_R}(a, b)$ special sets in Item 4 of Definition 2.3 that correspond to subsets of $N_{G_R}(u) \subseteq R$. For any $v \in M_b$ with $\{u, v\} \in \widetilde{G}_R$ (i.e., $N_{G_R}(u) \cap N_{G_R}(v) \neq \varnothing$), the multiplicity of the edge $\{u, v\}$ in C is at most the number of blue or red edges from u that land in any special set.

In light of this, we can utilize Item 1 of Definition 2.8 of the gadget, which states that the blue or red edges must be evenly spread among the special sets. For $u \in U_\ell \cap M_a$, suppose it has $\deg_S(u)$ edges to S on the left, then it has roughly $d_L \cdot \deg_S(u)$ blue or red edges going to R. Suppose u has λ neighbors in M_b for a fixed $b \in [k]$, which correspond to λ of the $s_R(a, b)$ special sets. Then, Item 1 of Definition 2.8 states that the number of blue or red edges that land in the these special sets is roughly as expected: $\lambda / s_R(a, b) \cdot d_L \cdot \deg_S(u)$. Moreover, by the small-set skeleton expansion of $\widetilde{G}_R$ (Definition 2.5), it is true that $\underline{C}[U_\ell]$, a subgraph of $\widetilde{G}_R[U_\ell]$, has average degree at most λ. This gives an upper bound on the total multiplicites of edges within U_ℓ in C.

Collisions between U_ℓ and U_h. Unfortunately, we no longer have the degree bounds for edges between U_ℓ and U_h in $\underline{C}$. For $u \in U_\ell \cap M_a$, if $\deg_{\underline{C}}(u \to U_h \cap M_b) \leqslant \lambda/\delta$ (for some $\delta = o_D(1)$ not too small), then we can still use the previous argument that only an $o_D(1)$ fraction of the blue or red edges land in the special sets. However, if $\deg_{\underline{C}}(u \to U_h \cap M_b)$ is large, we need a better argument.

We call such vertices *saturated*, denoted U_{sat}. We no longer have upper bounds using the edge multiplicities in C and the special sets. We must instead directly upper bound $|U_{\mathrm{sat}}|$. Consider the bipartite graph between saturated vertices $U_h \cap M_b$. The key observation is that the maximum eigenvalue is also upper bounded by λ, and since the degrees of the saturated vertices are $> \lambda/\delta$, the average degree of $U_h \cap M_b$ in this graph is $\lesssim \frac{\lambda^2}{\lambda/\delta} \leqslant \delta\lambda$ (Lemma 2.18). In particular, this implies that U_{sat} and $U_h \cap M_b$ are very unbalanced — $|U_{\mathrm{sat}}| \lesssim \delta^2 |U_h|$. On the other hand, since the vertices in U_h all have large degrees to S (in the graph G_L), we also have an upper bound on $|U_h|$ in terms of $e_{G_L}(S, U) = k|S|$. This completes the proof.

Concrete parameters and requirements. To prove the triangle expansion, we analyze the $(k-1)$-dimensional Ramanujan complex of [LSV05b, LSV05a] (see Section 3) and show an upper bound of $\tau|U|$ on the number of $(k-1)$-faces in the complex containing a triangle in any small vertex set U (see Lemma 3.11; the proof is carried out in Section 4). On the other hand, the skeleton expansion λ

follows from known spectral properties of the Ramanujan complex (see Lemma 3.12). The concrete bounds for D_L, τ and λ are stated in Lemma 2.7.

Remark 2.14 (Parameter requirements)**.** At a high level, we have two requirements. Let δ be some small enough constant. For the left-to-middle analysis, we need to set the degree threshold for $U_h \subseteq M$ to be τ/δ in order to get meaningful bounds on U_h. Thus, for the low-degree vertices to have good unique-neighbor expansion within each gadget H, we need $\frac{\tau}{\delta} \cdot d_L < D_L$. For the middle-to-right analysis, λ-expansion roughly means that any small subgraph has average degree λ, hence we need $d_L > \lambda/\delta$ (here, we ignore the multiplicities for simplicity). This gives

$$D_L/\tau \gg d_L \gg \lambda \,.$$

Unfortunately, the above requirements restrict us to $k \leqslant 5$. Let q be the large prime power used in the Ramanujan complex. From Lemma 2.7, we have:

- $k = 3$: $D_L = \Omega(q^3)$, $\tau = O(q^{1.5})$, and $\lambda = O(q)$. (See also Lemma 4.4.)
- $k = 5$: $D_L = \Omega(q^{10})$, $\tau = O(q^{6.5})$, and $\lambda = O(q^3)$. (See also Corollary 4.9.)
- $k = 6$: $D_L = \Omega(q^{15})$, $\lambda = O(q^{4.5})$, but the bound on τ we have is no better than $O(q^{10.5})$.

We can see that for $k = 6$, the parameters do not satisfy the aforementioned requirements.

2.3 Proof of Theorem 2.11

We first introduce some notation.

- Fix a subset $S \subseteq L$ of size $\leqslant \eta|L|$. Let $U \subseteq M$ be the neighbors of S in M in G_L, and let $\Gamma_S = G_L[S \cup U]$ be the induced subgraph of S, U.
- We use U_h ("high") to denote the set of vertices in U with degree (in Γ_S) exceeding $\tau \cdot \frac{1}{\delta}$, and we use U_ℓ ("low") to denote $U \setminus U_h$.
- For each $u \in U$, we draw blue edges from u to all its right neighbors that are unique-neighbors of $N_L(u)$ within the gadget H_u. We will use $\mathrm{Blue}(u)$ to refer to the blue edges incident to u.
- We draw red edges from u to all its right neighbors that are nonunique-neighbors of $N_L(u)$ within H_u. We will use $\mathrm{Red}(u)$ to refer to the red edges incident to u.
- We use $\widetilde{G}_R$ to denote the simple graph on vertex set M obtained by placing an edge for every $u, v \in M$ such that there is at least one length-2 walk (u, a, v) in G_R for $a \in R$.

We start with the left-to-middle analysis. The following lemma states that almost $\frac{k-2}{k}$ fraction of edges leaving S goes to the low-degree vertices U_ℓ.

Lemma 2.15. *Suppose $\delta k \leqslant 1/2$. The number of edges in Γ_S incident to U_ℓ is at least $(1-4\delta)(k-2)|S|$.*

Proof. By definition of U_h, the number of edges from S to U_h is at least $\tau \cdot \frac{|U_h|}{\delta}$. Define $S_{\geqslant 3}$ as the vertices of S with at least 3 neighbors into U_h. By the small-set triangle expansion (Definition 2.4) of G_L, the number of vertices in $S_{\geqslant 3}$ is at most $\tau \cdot |U_h|$. Since every vertex in $S_{\geqslant 3}$ is degree-k, we have $e(S_{\geqslant 3}, U_h) \leqslant k\tau|U_h|$. On the other hand, by definition of U_h, each vertex in U_h has at least τ/δ edges to S, so $e(S, U_h) \geqslant \tau|U_h|/\delta$. Consequently, we have

$$\frac{e(S_{\geqslant 3}, U_h)}{e(S, U_h)} \leqslant \delta k\,.$$

Every vertex in $S \setminus S_{\geqslant 3}$ has at most 2 edges into U_h, and hence at least $k-2$ edges into U_ℓ. Thus, we have:

$$e(S, U_\ell) \geqslant e(S \setminus S_{\geqslant 3}, U_\ell) \geqslant \frac{k-2}{2} e(S \setminus S_{\geqslant 3}, U_h) \geqslant \frac{k-2}{2}(1-\delta k) e(S, U_h)\,.$$

Since $\frac{b}{a+b}$ is monotone increasing in b for $a, b > 0$, we have:

$$\frac{e(S, U_\ell)}{k|S|} = \frac{e(S, U_\ell)}{e(S, U_\ell) + e(S, U_h)} \geqslant \frac{\frac{k-2}{2}(1-\delta k)}{\frac{k-2}{2}(1-\delta k) + 1} = \frac{k-2}{k}\left(1 - \frac{2\delta}{1-\delta(k-2)}\right),$$

which completes the proof. □

Lemma 2.16. *The number of vertices in R incident to exactly one blue edge and no red edges is at least $(1 - o_D(1))(k-2)d_L|S|$.*

Before we prove the above lemma, we first show how to complete the proof of Theorem 2.11.

Proof of Theorem 2.11. The statement is immediate from Observation 2.12 and Lemma 2.16. □

We will need the following folklore facts that we will apply to U_ℓ.

Lemma 2.17. *For a graph G, suppose λ is the maximum eigenvalue of the adjacency matrix, then there is an orientation of the edges in G such that all vertices have outdegree at most λ.*

Proof. Let A be the adjacency matrix of G, and let n be the number of vertices. Note that $\lambda_{\max}(A) \leqslant \lambda$ implies that all principal submatrices of A, i.e., induced subgraphs of G, have maximum eigenvalue at most λ. Moreover, $\lambda_{\max}(A) \leqslant \lambda$ implies that the average degree is at most λ, since $2|E(G)| = \vec{1}^\top A \vec{1} \leqslant \lambda \|\vec{1}\|_2^2 = \lambda n$. We can thus find a vertex v with degree at most λ and orient all its incident edges to point away from v. Then, we can remove v from G and repeat this process, since all induced subgraphs of G have average degree at most λ. In the end, we obtain an orientation of edges where all outdegrees are at most λ. □

Lemma 2.18 ([HMMP24, Lemma 6.2])**.** *Let G be a bipartite graph with average left-degree d_1 and average right-degree d_2. Let λ be the maximum eigenvalue of the adjacency matrix. Then, $(d_1 - 1)(d_2 - 1) \leqslant \lambda^2$.*

Now, we prove Lemma 2.16.

Proof of Lemma 2.16. We will refer to a vertex r in R incident to exactly one blue edge and no red edges as a *blue unique-neighbor*. By lossless expansion of the gadget H (Item 2 of Definition 2.8) and Lemma 2.15, the number of blue edges from U_ℓ to R is at least

$$\sum_{u \in U_\ell} (1 - o_D(1)) d_L \cdot \deg_{\Gamma_S}(u) = (1 - o_D(1)) d_L \cdot e(S, U_\ell) \geqslant (1 - o_D(1)) \cdot d_L (k-2)|S|\,,$$

where we use the fact that $\deg_{\Gamma_S}(u) \leqslant \frac{\tau}{\delta} \leqslant \frac{D}{d_L \log D}$ (required in Definition 2.8) for all $u \in U_\ell$.

To control the number of blue unique-neighbors, we bound the number of blue edges to R that "collide" with another edge using bounds on the edge density of small sets in $\widetilde{G}_R$.

Construct a (multi-)graph C on vertex set U by placing a copy of the edge $\{u, v\}$ for each $u \in U_\ell$, $v \in U$, $r \in R$ such that u has a blue edge to r and v has an (either blue or red) edge to r. Let $\underline{C}$ be the graph obtained by removing duplicate edges from C. By Observation 2.12, the number of blue unique-neighbors is at most

$$(1 - o_D(1)) \cdot (k-2) d_L |S| - 2e(C)\,.$$

It suffices to show that $e(C) \leqslant o_D(1) \cdot k d_L |S|$. To bound $e(C)$, we write it as $e_C(U_\ell) + e_C(U_\ell, U_h)$ (note that there are no edges within U_h in C), and bound each term separately.

We first bound $e(U_\ell)$. By λ-small-set skeleton expansion of $\widetilde{G}_R$, the largest eigenvalue of $\underline{C}[U_\ell]$ is bounded by λ. Consequently, by Lemma 2.17, there is an orientation of the edges of $\underline{C}[U_\ell]$ such that all vertices have outdegree bounded by λ. Pick such an orientation and let $O(v)$ denote the set of outgoing edges incident to a vertex v. We can write:

$$e_C(U_\ell) = \sum_{v \in U_\ell} \sum_{e \in O(v)} \text{multiplicity}(e)\,.$$

Fix $a \neq b \in [k]$. We will write $s := s_R(a, b)$ for simplicity (when $a, b \in [k]$ are clear from context). For $v \in M_a$, define $O_b(v)$ as the set of $e \in O(v)$ directed towards a vertex in M_b. By Observation 2.13, we can bound $\sum_{e \in O_b(v)} \text{multiplicity}(e)$ by the number of blue or red edges that land in any $|O_b(v)|$ of the special sets. Since $|O_b(v)| \leqslant \lambda < \frac{\lambda}{\delta}$ (here, we use a loose bound of $\frac{\lambda}{\delta}$ on $|O_b(v)|$ for convenience later), we can apply the bound in Item 1 of Definition 2.8 with $|S| = \deg_{\Gamma_S}(v)$ and $|W| = \max\{\frac{\lambda}{\delta}, \frac{s \log D}{d_L}\}$ and get

$$\begin{aligned}
\sum_{e \in O_b(v)} \text{multiplicity}(e) &\leqslant 32 \sum_{b=1}^{k} \max\left\{\frac{\lambda}{\delta}, \frac{s \log D}{d_L}\right\} \cdot \max\left\{\frac{1}{s} \cdot d_L \cdot \deg_{\Gamma_S}(v),\ \log D\right\} \\
&\leqslant 32 \sum_{b=1}^{k} \max\left\{\frac{\lambda}{\delta s}, \frac{\lambda \log D}{\delta d_L \deg_{\Gamma_S}(v)}, \frac{\log D}{d_L}, \frac{s \log^2 D}{d_L^2 \deg_{\Gamma_S}(v)}\right\} \cdot d_L \cdot \deg_{\Gamma_S}(v) \\
&\leqslant o_D(1) \cdot d_L \cdot \deg_{\Gamma_S}(v)\,.
\end{aligned} \tag{1}$$

Here, we use the assumptions on the parameters listed in Theorem 2.11: $\lambda \leqslant \delta^2 s$, and $d_L \geqslant \frac{1}{\delta} \max\{\lambda, \sqrt{s}\} \log^2 D \geqslant \log^2 D$.

Summing over all $b \in [k]$ gives at most a k factor, and $\sum_{v \in U_\ell} \deg_{\Gamma_S}(v) \leqslant k|S|$. Thus, we have

$$e_C(U_\ell) \leqslant o_D(1) \cdot k d_L |S|\,.$$

We now turn our attention to showing a bound on $e(U_\ell, U_h)$, which we can write as:

$$\sum_{a, b \in [k]} e_C(U_\ell \cap M_a, U_h \cap M_b)\,.$$

Again, we fix a, b and prove a bound on the summand. Call a vertex $v \in U_\ell \cap M_a$ *saturated* if $\deg_{\underline{C}}(v \to U_h \cap M_b) > \frac{\lambda}{\delta}$, and call a vertex *unsaturated* otherwise. We use U_{sat} and U_{unsat} to denote the saturated and unsaturated vertices in $U_\ell \cap M_a$ respectively.

For any unsaturated vertex, its degree in $\underline{C}$ is at most $\frac{\lambda}{\delta}$, and thus, the exact same calculation as Eq. (1) shows that

$$e_C(U_{\text{unsat}}, U_h \cap M_b) \leqslant o_D(1) \cdot kd_L|S| \,.$$

Now, we bound the contribution of saturated vertices. Here, we only have the naive bound

$$e_C(U_{\text{sat}}, U_h \cap M_b) \leqslant \sum_{u \in U_{\text{sat}}} \deg_{\Gamma_S}(u) \cdot d_L \leqslant \frac{\tau}{\delta} \cdot d_L |U_{\text{sat}}| \,,$$

since $\deg_{\Gamma_S}(u) \leqslant \frac{\tau}{\delta}$ for all $u \in U_{\text{sat}} \subseteq U_\ell$. It remains to bound $|U_{\text{sat}}|$.

Consider the bipartite graph between U_{sat} and $U_h \cap M_b$, within $\underline{C}$. We know that (i) the top eigenvalue of this bipartite graph is at most λ, by λ-small-set skeleton expansion of $\widetilde{G}_R$, and (ii) every vertex in U_{sat} has degree at least $\frac{\lambda}{\delta}$. Thus, by Lemma 2.18, the average degree of $U_h \cap M_b$ is at most $\delta\lambda + 1$ which is at most $2\delta\lambda$ by the assumption that $\delta\lambda \geqslant 1$. In particular, we have

$$|U_{\text{sat}}| \leqslant |U_h \cap M_b| \cdot \frac{2\delta\lambda}{\lambda/\delta} \leqslant 2\delta^2 |U_h| \,.$$

On the other hand, since every vertex u in U_h has $\deg_{\Gamma_S}(u) \geqslant \tau$, we have $\tau|U_h| \leqslant e_{\Gamma_S}(S, U_h) \leqslant k|S|$. This implies that

$$e_C(U_{\text{sat}}, U_h \cap M_b) \leqslant 2\delta \cdot kd_L|S| \leqslant o_D(1) \cdot kd_L|S| \,,$$

since $\delta \leqslant o_D(1)$. Summing up contributions from all $a, b \in [k]$ gives an extra k^2 factor.

Combining all of the above, we get that $e(C) \leqslant o_D(1) \cdot kd_L|S|$, completing the proof. □

3 Construction of the base graph

We first list the properties of the Ramanujan complex of [LSV05b, LSV05a] that we use to construct our base graph. For standard terminology pertaining simplicial complexes, see, e.g., [ALOV19, Section 2.4]. Moreover, we recall the definitions and approximations (for large q) of Gaussian binomial coefficients: $[k]_q = \frac{q^k-1}{q-1} \sim q^{k-1}$, $[k]_q! = [1]_q[2]_q \cdots [k]_q \sim q^{\binom{k}{2}}$, and $\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{[k]_q!}{[i]_q![k-i]_q!} \sim q^{i(k-i)}$, and note that $\begin{bmatrix} k \\ i \end{bmatrix}_q \leqslant \begin{bmatrix} k \\ \lfloor k/2 \rfloor \end{bmatrix}_q \sim q^{\lfloor k^2/4 \rfloor}$.

Definition 3.1 (Cayley clique complex)**.** Let Γ be a group, and let S be a symmetric set of generators of Γ. We use $\mathcal{H} = \mathrm{Cay}_k(\Gamma, S)$ to denote the $(k-1)$-dimensional complex constructed by choosing each $\{u_0, \dots, u_{k-1}\} \subseteq \Gamma$ as a face iff for every $a, b \in [k]$, $u_a u_b^{-1} \in S$. We refer to $\mathcal{H}$ as the *Cayley clique complex* on Γ with generating set S.

Definition 3.2 (Incidence graph)**.** The *incidence graph* $\mathrm{Inc}_{\mathcal{H}}$ of a simplicial complex $\mathcal{H}$ is a bipartite graph between the vertices and $(k-1)$-faces of $\mathcal{H}$, where there is an edge between a vertex v and a face e if e contains v.

Definition 3.3 (Unweighted skeleton)**.** The *unweighted skeleton* $\mathrm{Skel}_{\mathcal{H}}$ of a complex $\mathcal{H}$ refers to the simple graph with vertex set Γ obtained by placing an edge between u and v if they share a face.

Theorem 3.4 ([LSV05b, LSV05a])**.** *Fix a prime power q and integer $k \geqslant 2$. There is an algorithm that, on input an integer $e > 1$ (with $q^e > 4k^2 + 1$), constructs:*

- *The group $\Gamma = \mathrm{PGL}_k(\mathbb{F}_{q^e})$, which has cardinality $n := \frac{1}{q^e-1}\prod_{0\leqslant i\leqslant k}\left(q^{ek}-q^{ei}\right) \sim q^{ek^2-1}$.*

- *A set of generators $S = S_1 \sqcup \cdots \sqcup S_{k-1} \subset \Gamma$, where $|S_i| = \left[\begin{smallmatrix} k \\ i \end{smallmatrix}\right]_q \sim q^{i(k-i)}$ (the Gaussian binomial coefficient, equal to the number of subspaces of dimension k in $\mathbb{F}_q^k$); hence $|S| \sim q^{\lfloor k^2/4 \rfloor}$.*

The algorithm runs in time $\mathrm{poly}(n)$. The resulting Cayley complex $\mathrm{Cay}_k(\Gamma, S)$, called the Ramanujan clique complex, is a k-partite $(k-1)$-dimensional simplicial complex with the following properties.

- *The vertex set V, of size n, has equal-size parts $V_0, V_1, \ldots, V_{k-1}$.*

- *For each $0 < i < k$, the generating set S_i creates directed edges that go from V_a to V_{a+i} (all part indices are taken mod k).*

- *Let A_i denote the directed subgraph on V consisting of just those edges created by S_i, so A_i is out-regular of degree $\left[\begin{smallmatrix} k \\ i \end{smallmatrix}\right]_q \sim q^{i(k-i)}$.*

- *Thought of as adjacency matrices, $A_1, \ldots, A_{k-1}$ commute (and are normal); thus they are simultaneously diagonalizable. For each of the n eigenvectors, there is an "eigentuple" $\vec{\lambda}$ of eigenvalues $(\lambda_1, \ldots, \lambda_{k-1}) \in \mathbb{C}^{k-1}$ associated to $A_1, \ldots, A_{k-1}$.*

- *We say that an eigentuple is trivial if $|\lambda_i| = \left[\begin{smallmatrix} k \\ i \end{smallmatrix}\right]_q$ for all i. The eigenvectors corresponding to the trivial eigentuple take on a constant value on each part.*

- *Every nontrivial eigenvalue of A_k has modulus at most $\binom{k}{i}\sqrt{q^{i(k-i)}}$.*

- *For any i and any vertex $v \in V_i$, the link of v can be identified with the spherical building $\mathbb{P}(\mathbb{F}_q^k)$. Here, $\mathbb{P}(\mathbb{F}_q^k)$ is the $(k-1)$-partite $(k-2)$-dimensional simplicial complex whose vertices are all the non-trivial subspaces of $\mathbb{F}_q^k$ (i.e., not $\{0\}$ and not $\mathbb{F}_q^k$), and a t-face corresponds to a chain of subspaces $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_t$. The j-dimensional subspaces exactly correspond to the neighbors of v in V_{i+j}.*

We now use the known properties of the Ramanujan complex listed above in Theorem 3.4 to prove that for any pair of parts i and j, the bipartite graph of the 1-skeleton of the Ramanujan clique complex is $O(1)$-Ramanujan.

Lemma 3.5. *For $i, j \in [k]$ such that $i > j$, let A denote the adjacency matrix of the bipartite graph between V_i and V_j. We have:*

$$\lambda_2(A) \leqslant 2\binom{k}{i-j}\sqrt{q^{(i-j)(k-i+j)}}.$$

Proof. Let v be a nontrivial eigenvector of A. Observe that $(A_{i-j} + A_{j-i})v = (1 + \mathbf{1}[i-j = j-i])Av + w$, where the support of w is disjoint from that of Av. Thus: $\|Av\| \leqslant \left\|\left(A_{i-j} + A_{j-i}\right)v\right\|$.

Now, observe that for any $t \in [k]$, we have $\left\langle v_t, \vec{1}_t \right\rangle = 0$ where v_t and $\vec{1}_t$ denote their respective restrictions to V_t. Consequently, v must be orthogonal to all the trivial eigenvectors of $A_{i-j} + A_{j-i}$, and hence must be spanned by its nontrivial eigenvectors. By the bound on the nontrivial eigenvalues from Theorem 3.4, we have:

$$\left\|\left(A_{i-j} + A_{j-i}\right)v\right\| \leqslant 2\binom{k}{i-j}\sqrt{q^{(i-j)(k-i+j)}}.$$

Consequently, we have $\lambda_2(A) \leqslant 2\binom{k}{i-j}\sqrt{q^{(i-j)(k-i+j)}}$. □

3.1 Properties of the spherical building

Observation 3.6. The 1-skeleton of $\mathbb{P}(\mathbb{F}_q^k)$ is a $(k-1)$-partite graph with vertex sets $V_1, V_2, \ldots, V_{k-1}$, where V_i consists of i-dimensional subspaces in $\mathbb{F}_p^k$. Thus, $|V_i| = \left[{k \atop i}\right]_q$, which are the Gaussian binomial coefficients. For $i < j$, the bipartite graph between V_i and V_j in the 1-skeleton has left and right degree $\left[{k-i \atop j-i}\right]_q \approx q^{(j-i)(d-j)}$ and $\left[{j \atop i}\right]_q \approx q^{i(j-i)}$ respectively.

Remark 3.7. For example, when $k = 3$, we have $|V_1| = |V_2| = \left[{3 \atop 1}\right]_q = q^2 + q + 1$, and the bipartite graph between V_1 and V_2 has left and right degree $\left[{2 \atop 1}\right]_q = q + 1$.

Lemma 3.8 (Corollary 7.1 of [GHK⁺22])**.** *Let $1 \leqslant i < j \leqslant k-1$. The bipartite graph between V_i and V_j has second eigenvalue*

$$\lambda_2 = \sqrt{q^{j-i}\left[{j-1 \atop j-i}\right]_q \left[{k-i-1 \atop j-i}\right]_q} \approx q^{(j-i)(k+i-j-1)/2}.$$

3.2 Construction of base graph from Ramanujan complex: balanced case

We first specify how to construct a structured bipartite graph, which is immediately useful in the "balanced" setting.

Definition 3.9 (Construction of structured bipartite graph)**.** We describe a procedure that takes in nonnegative integers n_0, k, and prime power q, and for $n = \Theta(n_0)$ and $D = |\mathbb{P}(\mathbb{F}_q^k)|$, constructs vertex sets V, M, along with a (k, D)-biregular bipartite graph G on (V, M).

- We find e such that $n := \left|\mathrm{PGL}_k(\mathbb{F}_{q^e})\right|$ is $\Theta(n_0)$. Run the algorithm from Theorem 3.4 to construct the group $\Gamma = \mathrm{PGL}_k(\mathbb{F}_{q^e})$ along with generators $S = S_1 \sqcup \cdots \sqcup S_{k-1}$, closed under inverses.
- Let $M = \Gamma$, and let V be all the $(k-1)$-faces in the Cayley complex $\mathrm{Cay}_k(\Gamma, S)$, and define G as the graph where we place an edge between $m \in M$ and $v \in V$ if $m \in v$.
- Identify elements in $[D]$ with k-sets of the form $\{\mathrm{id}, s_1, \ldots, s_{k-1}\}$ where $s_i \in S_i$, and $s_a s_b^{-1} \in S$ for all $a, b \in [k]$. Here, id is the identity element of Γ. For each $m \in M$, we define Nbr_m as the function that maps $\{\mathrm{id}, s_1, \ldots, s_{k-1}\}$ to $\{m, ms_1, \ldots, ms_{k-1}\}$, which is a clique, and hence one of the right neighbors of m.

It remains to prove that the base graph constructed via the above procedure has useful properties.

Lemma 3.10. *The graphs G produced in Definition 3.9 is a structured bipartite graph in the sense of Definition 2.3 with parameters $D = [k]_q!$ and $s_G(a,b) = \left[{k \atop b-a}\right]_q \in [q^{k-1}, O(q^{\lfloor k^2/4 \rfloor})]$ for $a < b \in [k]$.*

Lemma 3.10 can be proved by mechanically verifying that G indeed satisfies all the claimed properties; we omit the details.

Lemma 3.11. *The graph G produced in Definition 3.9 is a τ-small-set triangle expander for*

$$\tau = O_k(1) \cdot q^{\binom{k}{2} - \frac{k^2}{8} - \frac{1}{2}} \cdot \max_{0 \leqslant i_0 < i_1 < i_2 < k} \; \min_{(i,j) \in \left\{ \substack{(i_1 - i_0, i_2 - i_0), \\ (i_2 - i_1, k + i_0 - i_1), \\ (k + i_0 - i_2, k + i_1 - i_2)} \right\}} q^{\frac{1}{8}((i-j+2)^2 + (k-i-j)^2)}.$$

We defer the proof of Lemma 3.11 to Section 4.

Lemma 3.12. *The graph G produced in Definition 3.9 is a λ-small-set skeleton expander for*

$$\lambda := O_k(1) \cdot q^{\frac{1}{2}\lfloor \frac{k^2}{4} \rfloor}\,.$$

Lemma 3.12 follows from Lemma 3.5 along with Lemma 3.13 below.

Lemma 3.13. *Let G be an n-vertex d-regular graph with largest nontrivial eigenvalue λ. Let H be any subgraph of G with edges incident to at most εn vertices. The largest eigenvalue of H is at most $\lambda + \varepsilon d$.*

Proof. Let v be the top eigenvector of A_H. Since A_H is a nonnegative matrix, by the Perron–Frobenius theorem, all entries of v are nonnegative. Further, the entries of v are 0 on isolated vertices in H, which ensures that v is supported on at most εn vertices. We have:

$$\lambda_{\max}(A_H) = v^\top A_H v \leqslant v^\top A_G v = v^\top \left(A_G - \frac{d}{n}11^\top\right) v + \frac{d}{n}\langle v, 1\rangle^2 \leqslant \lambda + \varepsilon d\,,$$

where the second step uses nonnegativity of v and $A_G - A_H$, and the final step uses Cauchy–Schwarz on v and $1_{\mathrm{supp}(v)}$. □

3.3 Imbalanced case

In the sequel, Γ and $S = S_1 \sqcup \cdots \sqcup S_{k-1}$ be as in Definition 3.9.

Definition 3.14 (Face generator). We call $\sigma = \{s_0 = \mathrm{id}, s_1, \ldots, s_{k-1}\}$ a *face generator* if $s_i \in S_i$, and for every $i, j \in [k]$, $s_i^{-1}s_j$ is in S. Given a collection F of face generators, we use $\mathrm{TruncCay}(\Gamma, F)$ to denote the complex obtained by including every face of the form $m\sigma$ for $m \in \Gamma$ and $\sigma \in F$.

Observation 3.15. Observe that for any collection F of face generators, the set of $(k-1)$-faces in the complex $\mathrm{TruncCay}(\Gamma, F)$ is a subset of the set of $(k-1)$-faces in $\mathrm{Cay}_k(\Gamma, S)$.

Definition 3.16 (Equivalence relation on face generators). We say face generators σ_1 and σ_2 are *equivalent* if for some $s \in S$, we have $\sigma_1 = s^{-1}\sigma_2$.

Observation 3.17. Any equivalence class can have at most k elements. This is because id is contained in both σ_1 and σ_2, and thus, for $\sigma_1 = s^{-1}\sigma_2$ to contain id, s must be in σ_2, for which there are only k choices.

Lemma 3.18. *There exists an integer $j \in [1, k]$ such that for any prime power q, and integer $D < \frac{1}{k}|\mathbb{P}(\mathbb{F}_q^k)| = \Theta\left(q^{\binom{k}{2}}\right)$ that is divisible by j, there is a set F of face generators of size exactly D such that the bipartite vertex-face incidence graph G of* $\mathrm{TruncCay}(\Gamma, F)$ *is a structured (k, D)-biregular graph, and further, G is a τ-small-set triangle expander and a λ-small-set skeleton expander for τ as in Lemma 3.11 and for λ as in Lemma 3.12.*

Proof. Define $\mathcal{F}(S)$ as the collection of all face generators on S. Observe that $\mathcal{F}(S)$ is in one-to-one correspondence with faces incident to any fixed $m \in \Gamma$, and hence $|\mathcal{F}(S)| = |\mathbb{P}(\mathbb{F}_q^k)|$. We construct F be starting with $\mathcal{F}(S)$ and deleting a subcollection of generators. First, partition $\mathcal{F}(S)$ into equivalence classes based on the equivalence relation in Definition 3.16. For some $1 \leqslant j \leqslant k$, at least $\frac{1}{k}|\mathcal{F}(S)|$ many face generators belong to an equivalence class of size j. We construct F of size $D < \frac{1}{k}|\mathcal{F}(S)|$ by choosing all the face generators from D/j arbitrary equivalence classes of size j.

We first prove that G is (k, D)-regular: observe that G is k-left-regular by definition. To see that it is D-right-regular, first observe that for any $m \in \Gamma$, m is contained in D faces of the form $m\sigma$ for $\sigma \in F$. We claim that there if a face f that contains m, then it must be of the form $m\sigma$ for some $\sigma \in F$. Observe that f must be of the form $m'\sigma'$ for some $m' \in \Gamma$ and face generator σ' in an equivalence class of size j to be included in the complex. Since $m'\sigma'$ contains m, $m'^{-1}m$ is in σ', and hence in S. Since S is closed under inverses, $m^{-1}m'$ is also in S. Now, we choose $\sigma = m^{-1}m'\sigma'$. We see that σ is a valid face generator since it contains id, and for any $a, b \in \sigma$, we have

$$a^{-1}b = a'^{-1}m'^{-1}mm^{-1}m'b' = a'^{-1}b' \in S.$$

Additionally, since $m^{-1}m$ is in S, σ and σ' belong to the same equivalence class, which implies that σ must be in F, since σ' is in F.

Finally, the claim that G is a structured bipartite graph in the sense of Definition 2.3 is a straightforward verification that we omit, and the small-set triangle and skeleton expansion properties follow from the fact that G is a subgraph of the graph constructed in Definition 3.9 combined with Lemmas 3.11 and 3.12. □

4 Bounding triangles

In this section, we prove Lemma 3.11. Fix a prime power q and integers $k, e \geqslant 2$. Let X denote the resulting Ramanujan clique complex as constructed in Theorem 3.4. Recall that we denote the vertex set of X by $X(0) = V_0 \cup V_1 \cup \cdots \cup V_{k-1}$, where V_i is the vertices in the i'th part. All V_i's have the same size, which we denote in this section by n. Let d_{ij} denote the degree of a left vertex in the bipartite graph (V_i, V_j) induced by the 1-skeleton of X. Recall that $d_{ij} = \begin{bmatrix} k \\ (j-i)_k \end{bmatrix}_q \approx q^{(j-i)_k(k-(j-i)_k)}$, where in general we use $a_k \in \{0, 1, \ldots, k-1\}$ to denote the number a reduced modulo k.

Our goal in this section is to upper bound the number of triangles contained in any small set $U \subseteq X(0)$. Our bounds follow from spectral bounds on edge density in the Ramanujan graph and in its links. The proof strategy loosely resembles the one taken in the proof of [DH24, Theorem 10.14].

4.1 Edge density of small sets in the Ramanujan complex

We will first prove some statements related to the edge density in small sets.

Lemma 4.1. *Let $\delta < \frac{1}{q^{k^2/4}}$ and $\alpha \in \mathbb{N}$. Let $U_{i,\alpha} \subseteq V_i$ and $U_j \subseteq V_j$ be such that $|U_{i,\alpha}| \leqslant \delta n$ and $|U_j| \leqslant \delta n$. Suppose also that the number of neighbors each vertex $u \in U_{i,\alpha}$ has in U_j is in $[2^{\alpha-1}, 2^{\alpha})$. Then,*

$$|U_{i,\alpha}| \leqslant O_k(1) \cdot q^{(j-i)_k(k-(j-i)_k)} \cdot 2^{-2\alpha} \cdot |U_j|.$$

We use the bipartite expander mixing lemma to prove the above statement.

Lemma 4.2 (Bipartite Expander-Mixing Lemma). *Let $G = (L, R, E)$ be a bipartite graph with left-degree c and right-degree d. Let $A \subseteq L$ have $|A|/|L| = \alpha$ and $B \subseteq R$ have $|B|/|R| = \beta$. Let λ denote the* *magnitude of the largest nontrivial (not $\pm\sqrt{cd}$) eigenvalue of G's adjacency matrix. Then*

$$\left|\frac{|E(A,B)|}{|E(L,R)|} - \alpha\beta\right| \leqslant \frac{\lambda}{\sqrt{cd}}\sqrt{\alpha(1-\alpha)\beta(1-\beta)} \leqslant \frac{\lambda}{\sqrt{cd}}\sqrt{\alpha\beta}\,. \tag{2}$$

Proof of Lemma 4.1. Consider the bipartite graph (V_i, V_j) induced by the 1-skeleton of X. Notice that both the left and right degrees are $d_{ij} = \begin{bmatrix} k \\ (j-i)_k \end{bmatrix}_q$. By Lemma 3.5, the second eigenvalue $\lambda_2(V_i, V_j)$ of this graph is bounded above by $O_k(1) \cdot \sqrt{q^{(j-i)_k(k-(j-i)_k)}}$.

We know that $e(U_{i,\alpha}, U_j) \geqslant 2^{\alpha-1} \cdot |U_{i,\alpha}|$. By the bipartite expander mixing lemma (Lemma 4.2), we have that

$$\begin{aligned} e(U_{i,\alpha}, U_j) &\leqslant \frac{|U_{i,\alpha}||U_j|d_{ij}}{n} + \lambda_2(V_i, V_j) \cdot \sqrt{|U_{i,\alpha}||U_j|} \\ &\leqslant \left(\delta d_{ij} + \lambda_2(V_i, V_j)\right) \cdot \sqrt{|U_{i,\alpha}||U_j|} \\ &= O_k(1) \cdot \sqrt{q^{(j-i)_k(k-(j-i)_k)}} \cdot \sqrt{|U_{i,\alpha}||U_j|}. \end{aligned}$$

Combining inequalities, this tells us that

$$2^{\alpha-1} \cdot |U_{i,\alpha}| \leqslant e(U_{i,\alpha}, U_j) \leqslant O_k(1) \cdot \sqrt{q^{(j-i)_k(k-(j-i)_k)}} \cdot \sqrt{|U_{i,\alpha}||U_j|},$$

which implies that

$$|U_{i,\alpha}| \leqslant O_k(1) \cdot q^{(j-i)_k(k-(j-i)_k)} \cdot 2^{-2\alpha} \cdot |U_j|\,.$$

□

The next lemma gives a bound on edge density between parts within the link of a vertex. For $u \in X(0)$, let $V_j(u)$ denote all the vertices in V_j that share an edge with u.

Lemma 4.3. *For $0 < i < j \in [k]$, and for $u \in V_0$, $U_i(u) \subseteq V_i(u)$, and $U_j(u) \subseteq V_j(u)$, it holds that*

$$e(U_i(u), U_j(u)) \leqslant O_k(1) \cdot \left(\frac{|U_i(u)||U_j(u)|}{q^{i(k-j)}} + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot \sqrt{|U_i(u)||U_j(u)|}\right).$$

Proof. Let L be the bipartite graph in the link on $(V_i(u), V_j(u))$. The left side has $d_{0i} = \begin{bmatrix} k \\ i \end{bmatrix}_q \approx q^{i(k-i)}$ vertices, and the right side has $d_{0j} = \begin{bmatrix} k \\ j \end{bmatrix}_q \approx q^{j(k-i)}$ vertices. The left degree is $\begin{bmatrix} k-i \\ k-j \end{bmatrix}_q \approx q^{(j-i)(k-j)}$ and the right degree is $\begin{bmatrix} j \\ i \end{bmatrix}_q \approx q^{i(j-i)}$.

By Lemma 3.8, the second eigenvalue of this graph is at most $\sqrt{q^{(j-i)(k-(j-i)-1)}}$. Thus, by the bipartite expander mixing lemma (Lemma 4.2), we have:

$$\begin{aligned} e(U_i(u), U_j(u)) &\leqslant \frac{|U_i(u)||U_j(u)|\sqrt{\begin{bmatrix} k-i \\ k-j \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q}}{\sqrt{\begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q}} + \lambda_2(V_i(u), V_j(u)) \cdot \sqrt{|U_i(u)||U_j(u)|} \\ &= O_k(1) \cdot \left(\frac{|U_i(u)||U_j(u)|}{q^{i(k-j)}} + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot \sqrt{|U_i(u)||U_j(u)|}\right). \end{aligned}$$

□

4.2 Warm up: bounding triangles in small sets in the 3-partite Ramanujan complex

Recall that our main goal of this section is to bound the number of triangles contained within small sets of vertices in the k-partite Ramanujan complex. As a warm-up, here we will calculate this quantity for the case of $k = 3$. The same ideas will generalize to the general case in Section 4.3, but we find it more transparent to first demonstrate this easier calculation.

In the case of $k = 3$, the vertices of the Ramanujan complex is of the form $X(0) = (V_0, V_1, V_2)$. Let $U = U_0 \cup U_1 \cup U_2 \subseteq X(0)$, where $U_i \subseteq V_i$ for $i \in [3]$, be a small set of vertices of X, satisfying that $|U| \leqslant \delta \cdot n = \frac{\delta}{3} \cdot |X(0)|$ where $\delta = \frac{1}{q^{9/4}}$ as to satisfy the conditions of Lemma 4.1.

The Ramanujan complex is a partite complex, with edges only between different parts, and hence any triangle contained within U must have exactly one vertex from each of U_0, U_1, and U_2. In fact, if we fix $u \in U_0$ and denote $U_1(u) := U_1 \cap V_1(u)$ and $U_2(u) := U_2 \cap V_2(u)$, then the number of triangles containing u is precisely the number of edges within the graph $(U_1(u), U_2(u))$. This latter quantity can be bounded by an application of the expander mixing lemma to the link of u (formally given in Lemma 4.3).

Our strategy for bounding the number of triangles contained within U can be described as follows:

(1) We will split the vertices in U_0 according to the number of neighbors it has within $U_1 \cup U_2$. For any $\alpha > 0$, Lemma 4.1 gives us an upper bound on the number of vertices $u \in U_0$ that can have $\approx 2^\alpha$ neighbors in $U_1 \cup U_2$.

(2) If a vertex $u \in U_0$ has $\approx 2^\alpha$ neighbors in $U_1 \cup U_2$, then the number of triangles containing u is equal to the number of edges in $(U_1(u), U_2(u))$ within the link of u. Lemma 4.3 gives us an upper bound on this quantity in terms of α.

We now execute this strategy. Let $\triangle$ denote the number of triangles contained within U.

Lemma 4.4. *Let $\delta < \frac{1}{q^{9/4}}$. For any $U = U_0 \cup U_1 \cup U_2 \subseteq X(0)$ with $|U| \leqslant \delta|X(0)|$, the number of triangles in $X(2)$ contained in U is at most $O(q^{3/2})|U|$.*

Proof. First, notice that for any $u \in U_0$ the number of neighbors it has within $V_1 \cup V_2$, and hence the maximal possible number of neighbors it has within $U_1 \cup U_2$, is upper bounded by $d_{01} + d_{02} = O(q^2)$. Let $B = \lceil \log_2(d_{01} + d_{02}) \rceil + 1 = O(\log_2 q)$.

We partition U_0 into sets $U_{0,\varnothing} \cup U_{0,1} \cup U_{0,2} \cup \cdots \cup U_{U_0,B}$ where

$$U_{0,\alpha} = \{u \in U_0 : |U_1(u)| + |U_2(u)| \in [2^{\alpha-1}, 2^\alpha)\}$$

and $U_{0,\varnothing}$ denotes all vertices in U_0 with no neighbors in $U_1 \cup U_2$. The vertices in $U_{0,\varnothing}$ contribute no triangles to our count, so we may ignore them. For any $\alpha \in [1, B]$, combining with the trivial observation that $|U_{0,\alpha}| \leqslant |U|$, Lemma 4.1 gives us an upper bound on the size of $U_{0,\alpha}$:

$$|U_{0,\alpha}| \leqslant \min\{1, O(q^2 2^{-2\alpha})\} \cdot |U| . \tag{3}$$

The number of triangles contained in U is equal to $\triangle = \sum_{u \in U_0} e(U_1(u), U_2(u))$. By Lemma 4.3, for any α and $u \in U_{0,\alpha}$, we have

$$e(U_1(u), U_2(u)) \leqslant O(1) \cdot \left(q^{-1} 2^{2\alpha} + \sqrt{q} \cdot 2^\alpha\right) .$$

Thus, we may upper bound the number of triangles in U by

$$\triangle \leqslant O(1) \cdot \sum_{\alpha=1}^{B} |U_{0,\alpha}| \cdot \left(q^{-1}2^{2\alpha} + \sqrt{q} \cdot 2^{\alpha}\right).$$

For $2^\alpha \leqslant q$, we have that $|U_{0,\alpha}| \leqslant |U|$. For $2^\alpha > q$, Equation (3) tells us that $|U_{0,\alpha}| \leqslant O(q^2 2^{-2\alpha})|U|$, in which case the summand $|U_{0,\alpha}| \cdot (q^{-1}2^{2\alpha} + \sqrt{q} \cdot 2^\alpha) \leqslant q + q^{5/2}2^{-\alpha}$. Combining the two, we get that the number of triangles in U is bounded by:

$$\begin{aligned}\triangle &\leqslant \sum_{\alpha:2^\alpha \leqslant q} \left(q^{-1}2^{2\alpha} + \sqrt{q} \cdot 2^\alpha\right) \cdot |U| + \sum_{\alpha:2^\alpha > q} \left(q + q^{5/2}2^{-\alpha}\right) \\ &= O(1) \cdot |U| \cdot \left(q^{-1} \cdot q^2 + \sqrt{q} \cdot q + q \log q + q^{5/2} \cdot q^{-1}\right) \\ &\leqslant O(q^{3/2})|U|,\end{aligned}$$

completing the proof. □

4.3 Bounding triangles in small sets: general case

We now bound the number of triangles within small sets of the k-partite Ramanujan complex. Our strategy is similar to the $k = 3$ case considered in the previous section but is more notationally complex.

Let $U = U_0 \cup U_1 \cup \cdots \cup U_{k-1} \subseteq X(0)$ be a small set of size $\leqslant \delta n = \frac{\delta}{k} \cdot |X(0)|$. Our goal is to bound the number of triangles contained within U.

Fix three parts $i_0 < i_1 < i_2 \in [k]$. Our goal is to bound the number of triangles contained within $U_{i_0} \cup U_{i_1} \cup U_{i_2}$ in terms of $|U|$, which we will denote by $\triangle_{i_0,i_1,i_2}$. In fact, by rotational symmetry of the Ramanujan complex, we may assume that $i_0 = 0$ (by replacing i_1 with $i_1 - i_0$ and i_2 with $i_2 - i_0$). So, let us relabel and consider the case of triangles within $U_0 \cup U_i \cup U_j$, where $0 < i < j < k$.

Lemma 4.5. *Let $\delta < \frac{1}{q^{k^2/4}}$. Let $U = U_0 \cup U_1 \cup \cdots \cup U_{k-1} \subseteq X(0)$ such that $|U| \leqslant \delta n$. For any $0 < i < j < k$, it holds that*

$$\triangle_{0,i,j} \leqslant O_k(1) \cdot \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \cdot |U|.$$

Proof. For any $u \in U_0$, note that $|V_i(u)| \leqslant d_{0i} < q^{k^2/4}$ and $|V_j(u)| \leqslant d_{0j} < q^{k^2/4}$. Let $B = \lceil \log_2 q^{k^2/4} \rceil = O_k(1) \cdot \log_2 q$.

Define $U_i(u) := U_i \cap V_i(u)$. We will partition U_0 into sets $U_{0,\varnothing} \cup (\bigcup_{0<\alpha,\beta\leqslant B} U_{0,\alpha,\beta})$, where $U_{0,\alpha,\beta} \subseteq U_0$ denotes the set of vertices $u \in U_0$ with $|U_i(u)| \in [2^{\alpha-1}, 2^\alpha)$ and $|U_j(u)| \in [2^{\beta-1}, 2^\beta)$, and $U_{0,\varnothing} \subseteq U_0$ denotes the set of vertices $u \in U_0$ with either $U_i(u) = \varnothing$ or $U_j(u) = \varnothing$. Notice that none of the $u \in U_{0,\varnothing}$ contribute any triangles.

By Lemma 4.1 and also observing that $|U_{0,\alpha,\beta}| \leqslant |U_0| \leqslant |U|$, we have that

$$|U_{0,\alpha,\beta}| \leqslant O_k(1) \cdot \min\left\{1, q^{i(k-i)} \cdot 2^{-2\alpha}, q^{j(k-j)} \cdot 2^{-2\beta}\right\} \cdot |U|.$$

For any $u \in U_{0,\alpha,\beta}$, we know that $|U_i(u)| \in [2^{\alpha-1}, 2^\alpha)$ and $|U_j(u)| \in [2^{\beta-1}, 2^\beta)$. Thus, by Lemma 4.3 we have:

$$e(U_i(u), U_j(u)) \leqslant O_k(1) \cdot \left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}} + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot 2^{(\alpha+\beta)/2}\right).$$

Altogether, this gives us that

$$
\begin{aligned}
\triangle_{0,i,j} &= \sum_{u\in U_0} e(U_i(u),U_j(u)) \\
&\leqslant \sum_{0<\alpha,\beta\leqslant B} |U_{0,\alpha,\beta}|\cdot O_k(1)\cdot\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
&\leqslant O_k(|U|)\cdot\sum_{0<\alpha,\beta\leqslant B}\min\{1,q^{i(k-i)}\cdot 2^{-2\alpha},q^{j(k-j)}\cdot 2^{-2\beta}\}\cdot\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
&= O_k(|U|)\cdot(\mathsf{term1}+\mathsf{term2}+\mathsf{term3})\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathsf{term1} &= \sum_{\substack{0<\alpha\leqslant\frac{i(k-i)}{2}\log_2 q\\ 0<\beta\leqslant\frac{j(k-j)}{2}\log_2 q}}\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
\mathsf{term2} &= \sum_{\substack{\frac{i(k-i)}{2}\log_2 q<\alpha\leqslant B\\ 0<\beta\leqslant\alpha+\frac{j(k-j)-i(k-i)}{2}\log_2 q}} q^{i(k-i)}\cdot 2^{-2\alpha}\cdot\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
\mathsf{term3} &= \sum_{\substack{\frac{j(k-j)}{2}\log_2 q<\beta\leqslant B\\ 0<\alpha\leqslant\beta+\frac{i(k-i)-j(k-j)}{2}\log_2 q}} q^{j(k-j)}\cdot 2^{-2\beta}\cdot\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right).
\end{aligned}
$$

We bound each term separately. For term1, by separately evaluating the sum over β, then α, we have that

$$
\begin{aligned}
\mathsf{term1} &= \sum_{\substack{0<\alpha\leqslant\frac{i(k-i)}{2}\log_2 q\\ 0<\beta\leqslant\frac{j(k-j)}{2}\log_2 q}}\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
&= \sum_{0<\alpha\leqslant\frac{i(k-i)}{2}\log_2 q} O(1)\cdot\left(\frac{2^{\alpha}\cdot q^{j(k-j)/2}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{\alpha/2}\cdot 2^{j(k-j)/4}\right) \\
&= O(1)\cdot\left(\frac{q^{i(k-i)/2}\cdot q^{j(k-j)/2}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot q^{i(k-i)/4}\cdot q^{j(k-j)/4}\right) \\
&= O(1)\cdot\left(q^{(j-i)(k-(j-i))/2}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot q^{i(k-i)/4}\cdot q^{j(k-j)/4}\right).
\end{aligned}
$$

Next, for term2, we first sum over β, then over α, getting

$$
\begin{aligned}
\mathsf{term2} &= \sum_{\substack{\frac{i(k-i)}{2}\log_2 q<\alpha\leqslant B\\ 0<\beta\leqslant\alpha+\frac{j(k-j)-i(k-i)}{2}\log_2 q}} q^{i(k-i)}\cdot 2^{-2\alpha}\cdot\left(\frac{2^{\alpha+\beta}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{(\alpha+\beta)/2}\right) \\
&= O(1)\cdot\sum_{\frac{i(k-i)}{2}\log_2 q<\alpha<B} q^{i(k-i)}\cdot 2^{-2\alpha}\cdot\left(\frac{2^{2\alpha}\cdot q^{(j(k-j)-i(k-i))/2}}{q^{i(k-j)}}+\sqrt{q^{(j-i)(k-(j-i)-1)}}\cdot 2^{\alpha}\cdot q^{(j(k-j)-i(k-i))/4}\right)
\end{aligned}
$$

$$= O(1) \cdot \sum_{\frac{i(k-i)}{2}\log_2 q < \alpha < B} \left(q^{(j-i)(k-(j-i))/2} + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot 2^{-\alpha} \cdot q^{(j(k-j)+3i(k-i))/4} \right)$$
$$= O_k(1) \cdot \left(q^{(j-i)(k-(j-i))/2} \cdot \log_2 q + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i))/4} \cdot q^{(j(k-j)/4} \right).$$

Similarly, for term3, we first sum over α then β, and get

$$\mathsf{term3} \leqslant O_k(1) \cdot \left(q^{(j-i)(k-(j-i))/2} \cdot \log_2 q + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \right).$$

Now, notice that all three terms are of the form

$$O_k(1) \cdot \left(q^{(j-i)(k-(j-i))/2} \cdot \log_2 q + \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \right).$$

In fact, it always holds that

$$q^{(j-i)(k-(j-i))/2} \cdot \log_2 q = o\left(\sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \right), \tag{4}$$

so all three terms are

$$O_k(1) \cdot \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4},$$

so

$$\triangle_{0,i,j} \leqslant O_k(1) \cdot \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \cdot |U|.$$

Finally, to see (4), we divide both sides by $q^{(j-i)(k-(j-i))/2}$, which gives

$$\log_2 q \leqslant o(1) \cdot q^{-(j-i)/2} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} = o(1) \cdot q^{(i(k-i+2)+j(k-j-2))/4}.$$

Since $1 \leqslant i < j \leqslant k-1$, it follows that $i(k-i+2)$ is minimized at $i=1$, and $j(k-j-2) < 0$ only if $j = k-1$.

$$\frac{1}{4}(i(k-i+2) + j(k-j-2)) \geqslant \frac{1}{4}((k+1)-(k-1)) = \frac{1}{2}.$$

This implies that

$$\frac{(j-i)(k-(j-i))}{2} + \frac{1}{2} \leqslant \frac{(j-i)(k-(j-i)-1)}{2} + \frac{i(k-i)}{4} + \frac{j(k-j)}{4},$$

thus establishing (4). □

It follows by the rotational symmetry of the Ramanujan complex that a similar bound holds for any choice of three parts.

Corollary 4.6. *Let $\delta < \frac{1}{q^{k^2/4}}$. Let $U = U_0 \cup U_1 \cup \cdots \cup U_{k-1} \subseteq X(0)$ such that $|U| \leqslant \delta n$. For any $i_0 < i_1 < i_2 \in [k]$, it holds that*

$$\triangle_{i_0,i_1,i_2} \leqslant O_k(1) \cdot \min_{(i,j) \in \left\{ \begin{smallmatrix} (i_1-i_0, i_2-i_0), \\ (i_2-i_1, k+i_0-i_1), \\ (k+i_0-i_2, k+i_1-i_2) \end{smallmatrix} \right\}} \left[\sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \cdot |U| \right].$$

4.4 Bounding faces with triangles in small sets

In this subsection, we bound the number of $(k-1)$-faces of X that contain a triangle within a small vertex set U. We let this set of $(k-1)$-faces be denoted by $F^{k,3}(U)$. Formally, we define:

$$F^{k,3}(U) = \{f \in X(k-1) : |f \cap U| \geqslant 3\}.$$

To bound $|F^{k,3}(U)|$, we will bound for each $i_0 \neq i_1 \neq i_2 \in [k]$ the number Δ_{i_0,i_1,i_2} of triangles contained within $U_{i_0} \cup U_{i_1} \cup U_{i_2}$, then multiply by the number of ways to extend each triangle to a $(k-1)$-face. We've already given an upper bound on Δ_{i_0,i_1,i_2} in Corollary 4.6. The second quantity is given in the following claim.

Claim 4.7. *For $0 \leqslant i_0 < i_1 < i_2 < k$, and for $v_{i_0} \in V_{i_0}$, $v_{i_1} \in V_{i_1}$, and $v_{i_2} \in V_{i_2}$ such that $\{v_{i_0}, v_{i_1}, v_{i_2}\} \in X(2)$, there are $O_k(1) \cdot q^{\binom{i_1-i_0}{2}+\binom{i_2-i_1}{2}+\binom{k+i_0-i_2}{2}}$ $(k-1)$-faces containing $\{v_{i_0}, v_{i_1}, v_{i_2}\}$.*

Proof. Let us set $i_1' = i_1 - i_0$ and $i_2' = i_2 - i_0$, and look within the link of v_{i_0}. The vertices v_{i_1} and v_{i_2} correspond to a i_1'-plane $\rho_{i_1'}$ and i_2'-plane $\rho_{i_2'}$ within $\mathbb{P}\left(\mathbb{F}_q^k\right)$. The quantity we are interested in is the number of ways to choose a sequence of planes $\rho_1 \subseteq \rho_2 \subseteq \cdots \subseteq \rho_{k-1}$ with $\rho_{i_1'}$ and $\rho_{i_2'}$ fixed, which is equal to

$$\left(\begin{bmatrix} i_1' \\ 1 \end{bmatrix}_q \cdot \frac{\begin{bmatrix} i_1' \\ 2 \end{bmatrix}_q}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q} \cdot \cdots \cdot \frac{\begin{bmatrix} i_1' \\ i_1'-1 \end{bmatrix}_q}{\begin{bmatrix} i_1'-1 \\ i_1'-2 \end{bmatrix}_q}\right) \cdot \left(\frac{\begin{bmatrix} i_2' \\ i_1'+1 \end{bmatrix}_q}{\begin{bmatrix} i_1'+1 \\ i_1' \end{bmatrix}_q} \cdot \cdots \cdot \frac{\begin{bmatrix} i_2' \\ i_2'-1 \end{bmatrix}_q}{\begin{bmatrix} i_2'-1 \\ i_2'-2 \end{bmatrix}_q}\right) \cdot \left(\frac{\begin{bmatrix} k \\ i_2'+1 \end{bmatrix}_q}{\begin{bmatrix} i_2'+1 \\ i_2' \end{bmatrix}_q} \cdot \cdots \cdot \frac{\begin{bmatrix} k \\ k-1 \end{bmatrix}_q}{\begin{bmatrix} k-1 \\ k-2 \end{bmatrix}_q}\right),$$

which is $O_k(1) \cdot q^{\binom{i_1'}{2}+\binom{i_2'-i_1'}{2}+\binom{k-i_2'}{2}} = O_k(1) \cdot q^{\binom{i_1-i_0}{2}+\binom{i_2-i_1}{2}+\binom{k+i_0-i_2}{2}}$. □

Combining Corollary 4.6 and Claim 4.7, we obtain the following bound on $F^{k,3}(U)$.

Lemma 4.8. *Let $\delta < \frac{1}{q^{k^2/4}}$. Let $U = U_0 \cup U_1 \cup \cdots \cup U_{k-1} \subseteq X(0)$ such that $|U| \leqslant \delta n$. Then,*

$$|F^{k,3}(U)| \leqslant O_k(1) \cdot q^{\binom{k}{2}-\frac{k^2}{8}-\frac{1}{2}} \cdot \max_{0 \leqslant i_0 < i_1 < i_2 < k} \min_{(i,j) \in \left\{\begin{matrix} (i_1-i_0, i_2-i_0), \\ (i_2-i_1, k+i_0-i_1), \\ (k+i_0-i_2, k+i_1-i_2) \end{matrix}\right\}} q^{\frac{1}{8}((i-j+2)^2+(k-i-j)^2)} \cdot |U|.$$

Proof. Combining Corollary 4.6 and Claim 4.7, we get an upper bound of

$$O_k(1) \cdot \sqrt{q^{(j-i)(k-(j-i)-1)}} \cdot q^{i(k-i)/4} \cdot q^{j(k-j)/4} \cdot q^{\binom{i}{2}+\binom{j-i}{2}+\binom{k-j}{2}} \cdot |U|,$$

where $i < j$ are the indices obtained from taking the maximum over $i_0 < i_1 < i_2 < k$ in Claim 4.7 and the minimum as in Corollary 4.6. It is a straightforward calculation that this simplifies to the expression in the lemma statement. □

4.5 The case of $k = 5$

Finally, we apply our bounds from this section to the specific case of $k = 5$, which we will use in our construction of two-sided unique neighbor expanders.

Corollary 4.9. *Let $k = 5$, and let $\delta < q^{-25/4}$. Let $U = U_0 \cup U_1 \cup U_2 \cup U_3 \cup U_4 \subseteq X(0)$ such that $|U| \leqslant \delta n$. Then,*

$$|F^{5,3}(U)| \leqslant O(q^{13/2} \cdot |U|) .$$

Proof. By Lemma 4.8, we would like to evaluate the maximum over all tuples $0 \leqslant i_0 < i_1 < i_2 < 5$ of

$$\min_{(i,j) \in \left\{ \begin{smallmatrix} (i_1 - i_0, i_2 - i_0), \\ (i_2 - i_1, k + i_0 - i_1), \\ (k + i_0 - i_2, k + i_1 - i_2) \end{smallmatrix} \right\}} O_k(1) \cdot q^{\binom{k}{2} - \frac{k^2}{8} - \frac{1}{2}} \cdot q^{\frac{1}{8}((i-j+2)^2 + (k-i-j)^2)} \cdot |U| . \tag{5}$$

Let us perform casework on the value of (i_0, i_1, i_2). Because the Ramanujan complex is rotationally symmetric, there are essentially two cases to consider: the case where i_0, i_1, i_2 are consecutive indices, or the case where they are not all consecutive. The first case is equivalent to the case of $0, 1, 4$, and the second case is equivalent to the case of $0, 2, 3$.

In the first case, where $i_0 = 0$, $i_1 = 1$, and $i_2 = 4$, we have that Equation (5) is at most

$$O(1) \cdot q^{3/2} \cdot q^{4/4} \cdot q^{4/4} \cdot q^3 \cdot |U| = O(1) \cdot q^{13/2} \cdot |U| .$$

In the second case, where $i_0 = 0$, $i_1 = 2$, and $i_2 = 3$, we have that Equation (5) is at most

$$O(1) \cdot q^{3/2} \cdot q^{3/2} \cdot q^{3/2} \cdot q^2 \cdot |U| = O(1) \cdot q^{13/2} \cdot |U| .$$ □

5 Random gadget analysis

In this section, we prove Lemma 2.10 which states that there exist bipartite graphs H such that H and $H^\top$ both satisfy the properties in Definition 2.8.

Definition (Restatement of Definition 2.8)**.** Let H be a (d_L, d_R)-biregular bipartite graph on vertex set $([D_L], [D_R])$. For each $a, b \in [k]$, let $(A_i \subseteq [D_R])_{i \in [s]}$ be the *special sets* from Definition 2.3 where $s = s_R(a, b)$. Define $D := D_L + D_R$. We say H is a *pseudorandom gadget* if for every $a, b \in [k]$, we have the following properties:

1. For every $S \subseteq [D_L]$ such that $|S| \leqslant D_R/d_L$ and for every $W \subseteq [s]$ with $|W| \geqslant \frac{s \log D}{d_L}$,
$$\sum_{i \in W} |N(S) \cap A_i| \leqslant 32|W| \cdot \max\left\{ \frac{1}{r} \cdot d_L|S|, \log D \right\} .$$
2. For any $S \subseteq [D_L]$ with $|S| = o_D(1) \cdot D_R/d_L$, we have $|N(S)| \geqslant (1 - o_D(1))d_L|S|$.

In fact, we will prove that a random one satisfies the properties with high probability. The desired statement then follows since H and $H^\top$ have the same distribution. Throughout this section, we will write random variables in **boldface**.

Lemma 5.1 (Consequence of Lemma B.1, B.2 of [HMMP24])**.** *Let $\boldsymbol{H}$ be a random (d_1, d_2)-biregular graph on vertex sets $A \cup B$, and let $n_1 = |A|$, $n_2 = |B|$, $n = n_1 + n_2$, and $p = d_2/n_1$ (observe that $d_2/n_1 = d_1/n_2$). Suppose $p \leqslant o_n(1)$ and $d_1, d_2 \geqslant \log^2 n$. Then with probability $1 - o_n(1)$, $\boldsymbol{H}$ satisfies the following properties:*

- *For every non-empty $S \subseteq A$, $\frac{|\mathrm{UN}_{\boldsymbol{H}}(S)|}{|S|} \geqslant d_1(1-p)^{|S|-1} - \sqrt{4p(1-p)^{|S|-1} n_1 \log n_1} - o_n(d_1)$.*
- *For every non-empty $S \subseteq B$, $\frac{|\mathrm{UN}_{\boldsymbol{H}}(S)|}{|S|} \geqslant d_2(1-p)^{|S|-1} - \sqrt{4p(1-p)^{|S|-1} n_2 \log n_2} - o_n(d_2)$.*

Note that Lemma 5.1 implies that a random biregular graph satisfies property (2) in Definition 2.8. We now proceed to prove property (1).

We first state the following standard concentration bound, which was also used in [HMMP24].

Lemma 5.2 (Concentration for sampling without replacement)**.** *Fix $1 \leqslant \ell \leqslant n$. Let $S \subseteq [n]$, let $\boldsymbol{T}$ be a random sample of ℓ elements from $[n]$ without replacement, and let $\mu = \frac{\ell}{n}|S|$. Then, for all $\delta > 0$,*

$$\mathbf{Pr}[|S \cap \boldsymbol{T}| \geqslant (1+\delta)\mu] \leqslant \exp\left(-\frac{\delta^2 \mu}{2+\delta}\right).$$

We now prove that a random biregular graph satisfies property (1) of Definition 2.8.

Lemma 5.3. *Let $\boldsymbol{H}$ be a random (d_1, d_2)-biregular graph on vertex sets $A \cup B$, and let $n_1 = |A|$, $n_2 = |B|$ and $n = n_1 + n_2$. Let $B = B_1 \cup \cdots \cup B_r$ be a fixed partition of B such that $\frac{n_2}{2r} \leqslant |B_i| \leqslant \frac{2n_2}{r}$ for each $i \in [r]$. Then, with probability $1 - O(1/n)$, we have that for all $S \subseteq A$ with $|S| \leqslant n_2/d_1$ and all $W \subseteq [r]$ with $|W| \geqslant \frac{r \log n}{d_1}$,*

$$\sum_{i \in W} |N(S) \cap B_i| \leqslant 32|W| \cdot \max\left\{\frac{d_1}{r}|S|,\ \log n\right\}.$$

Proof. Consider a fixed $S \subseteq A$ of size $s \leqslant n_2/d_1$ and $W \subseteq [r]$ of size $w \geqslant \frac{r \log n}{d_1}$. Let $\ell := \sum_{i \in W} |B_i|$. By symmetry, the distribution of $\boldsymbol{H}$ is the same if we permute the vertices in B. Thus, the random variable $\sum_{i \in W} |N_{\boldsymbol{H}}(S) \cap B_i|$ has the same distribution as $|N_{\boldsymbol{H}}(S) \cap \boldsymbol{T}|$ where $\boldsymbol{T} \subseteq B$ is a uniformly random subset of size ℓ, independent of $\boldsymbol{H}$.

Since $N_H(S) \leqslant d_1 s$ for any H (with probability 1), we can instead analyze the tail probabilities of $|U \cap \boldsymbol{T}|$ for any fixed $U \subseteq B$ of size $d_1 s$. We have that $\mathbf{Pr}[|N_{\boldsymbol{H}}(S) \cap \boldsymbol{T}| \geqslant \lambda] \leqslant \mathbf{Pr}[|U \cap \boldsymbol{T}| \geqslant \lambda]$.

The random set $\boldsymbol{T}$ can be viewed as ℓ samples from B *without* replacement. By Lemma 5.2, we get the same concentration bounds as if it is a sum of ℓ samples with replacement: letting $\mu := \mathbf{E}[|U \cap \boldsymbol{T}|] = d_1 s \ell / n_2$, for any $\delta > 0$,

$$\mathbf{Pr}[|U \cap \boldsymbol{T}| \geqslant (1+\delta)\mu] \leqslant \exp\left(-\frac{\delta^2 \mu}{2+\delta}\right).$$

By assumption we have $w \cdot \frac{n_2}{2r} \leqslant \ell \leqslant w \cdot \frac{2n_2}{r}$. Thus, it follows that $d_1 s \frac{w}{2r} \leqslant \mu \leqslant d_1 s \frac{2w}{r}$. We now split into two cases depending on the size of s. Let the threshold be $\tau := \frac{r \log n}{d_1}$, which corresponds to $\mu \approx w \log n$. Let $C = 8$.

For $s \geqslant \tau$, we set $\delta = 2C - 1$, and we have that $\mathbf{Pr}[|U \cap \boldsymbol{T}| \geqslant 2C\mu] \leqslant \exp(-C\mu) \leqslant \exp(-C d_1 s \frac{w}{2r})$. In this case, $2C\mu \leqslant 4C d_1 s \frac{w}{r}$.

For $s \leqslant \tau$, we set δ such that $(1+\delta)\mu = 2Cw \log n$. In this case, we have $\frac{2Cw \log n}{\mu} \geqslant \frac{Cr \log n}{d_1 s} \geqslant C \geqslant 5$, which means that $\delta \geqslant 4$ and hence $\frac{\delta^2}{2+\delta} \geqslant \frac{1}{2}(1+\delta)$. Thus, $\mathbf{Pr}[|U \cap \boldsymbol{T}| \geqslant 2Cw \log n] \leqslant \exp(-Cw \log n)$.

We next union bound over all $S \subseteq A$ of size s.

$$\sum_{s=1}^{\tau-1} \binom{n_1}{s} \cdot e^{-Cw\log n} + \sum_{s=\tau}^{n_2/d_1} \binom{n_1}{s} e^{-Cd_1 s \frac{w}{2r}} \leqslant e^{\tau \log n_1 - Cw \log n} + \sum_{s=\tau}^{n_2/d_1} e^{s(\log n_1 - \frac{Cd_1 w}{2r})}.$$

Since we assume $w \geqslant \frac{r \log n}{d_1} = \tau$, for $C \geqslant 8$ we have $\frac{Cd_1 w}{2r} \geqslant \frac{C}{2} \log n \geqslant 4 \log n_1$. Thus, we can bound the above by $e^{-7w\log n} + e^{-\tau \cdot \frac{3d_1 w}{r}} \leqslant e^{-2w \log n}$.

Then, we union bound over $W \subseteq [r]$, which has at most $r^w \leqslant e^{w \log n}$ choices. Thus, with probability $1 - O(1/n)$, we have

$$\sum_{i \in W} |N_{\boldsymbol{H}}(S) \cap B_i| \leqslant \max\left\{4Cd_1 s \frac{w}{r},\, 2Cw \log n\right\} \leqslant 32w \cdot \max\left\{\frac{d_1 s}{r},\, \log n\right\}. \qquad \square$$

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